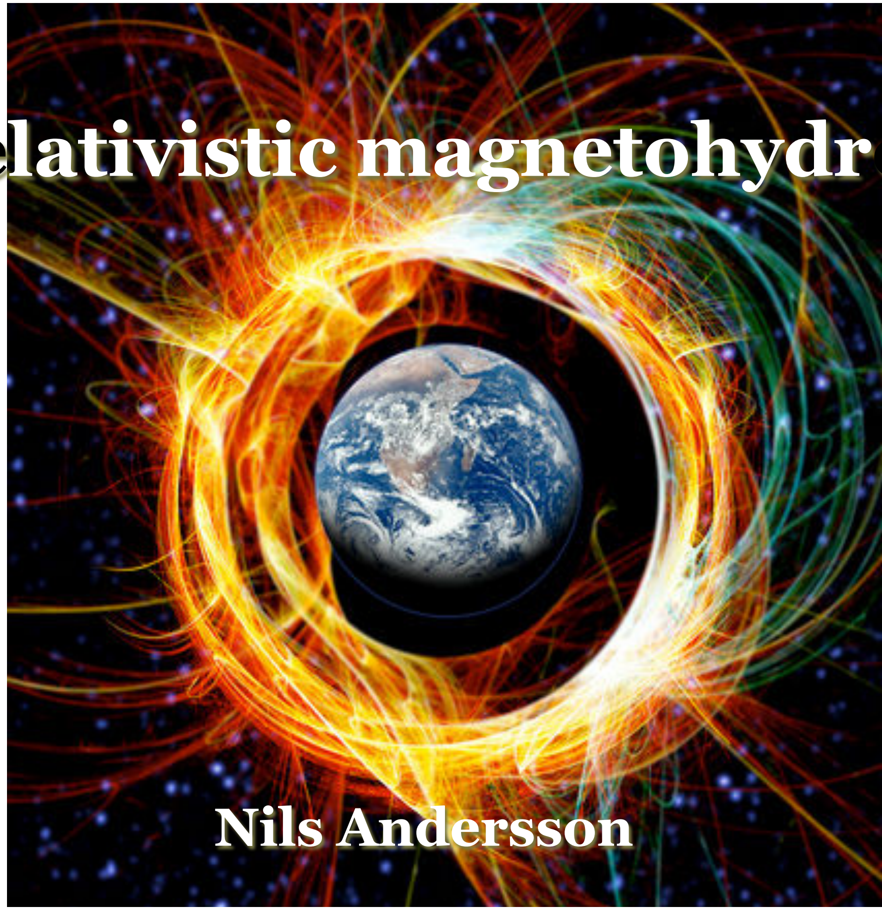
Resistive relativistic magnetohydrodynamics



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context

Relativistic fluid dynamics can be applied to a range of problems in astrophysics and high-energy physics.

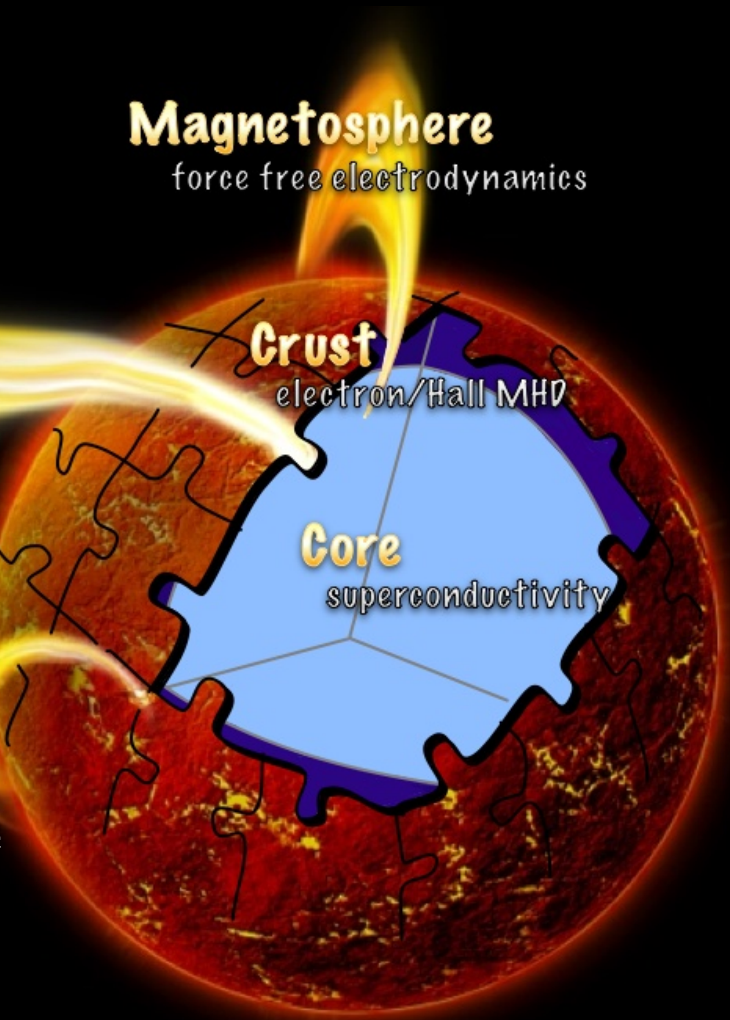
General Relativity is key to problems where the influence of gravity cannot be ignored;

- black holes
- neutron stars
- cosmology

Covariant variational framework provides a conceptually elegant description that helps understanding of complex systems.

Electromagnetism is easily incorporated in the formalism.

A number of interesting problems requires resistivity.



variational approach

Maxwell's equations can be obtained by varying the action

$$I_{\rm EM} = \int L_{\rm EM}\sqrt{-g}\ d^4x$$

where the Lagrangian is (for a "passive" medium)

$$L_{\rm EM} = -\frac{1}{4\mu_0}F_{ab}F^{ab} + j^a A_a$$

where j^a is the (conserved) charge current, and the (anti-symmetric) Faraday tensor given by;

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$

The variation is taken with respect to the vector potential A_a.

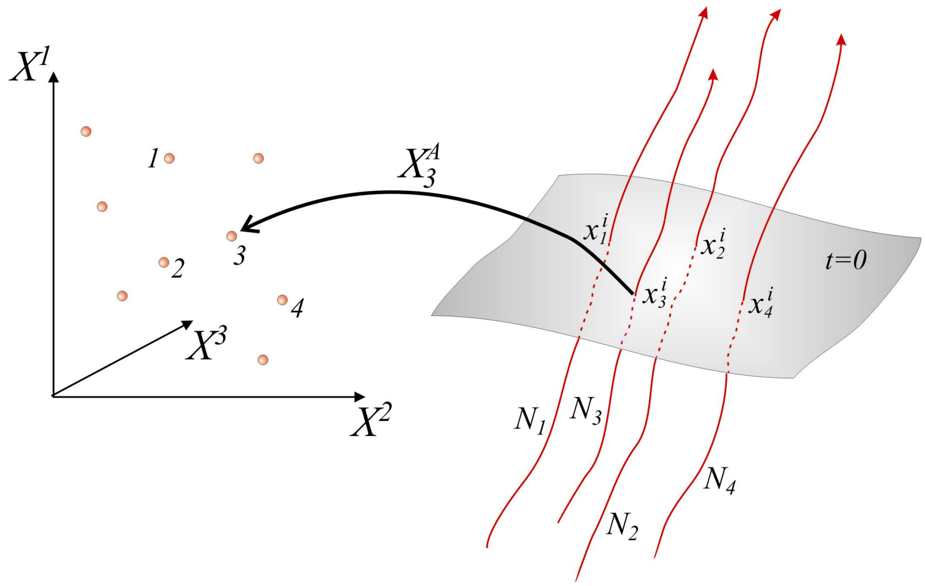
Note: The physical fields depend on the observer.

Question: How do you couple the electromagnetic field to a charged fluid?

convective variations

Make use of the convective variational approach developed by Carter and collaborators. Consider a single (uncharged) fluid as an example. Then we have the action;

$$I_{\rm M} = \int \Lambda \sqrt{-g}\ d^4x$$



where $\Lambda=\Lambda(n)$ and the conjugate momentum μ_a is determined by the variation

$$\delta\Lambda = \mu_a \delta n^a + \frac{1}{2} n^a \mu^b \delta g_{ab}$$

The variation is constrained by focussing on displacements ξ^a that keeps the flux n^a conserved. This leads to

$$\delta n^a = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a \left(\nabla_b \xi^b + \frac{1}{2} g^{bc} \delta g_{bc} \right)$$

Rearrange the variation and neglect “surface terms” to arrive at the final result.

$$\delta\left(\Lambda\sqrt{-g}\right) = \sqrt{-g}\left[2\xi^a n^b \nabla_{[a}\mu_{b]} + \frac{1}{2}\left(\Psi g^{ab} + n^a \mu^b\right)\delta g_{ab}\right]$$

$$2n^b \nabla_{[a}\mu_{b]} = 0$$

$$T^{ab} = \Psi g^{ab} + n^a \mu^b$$

$$\Psi = \Lambda - n^a \mu_a$$

When the Euler equation is satisfied then it is **automatically** true that the divergence of the stress-energy tensor vanishes.

a comfortable marriage

It is (relatively) straightforward to extend the constrained variation to charged fluids.

Let us consider the general case with a number of (conserved) fluxes carrying charges. Then we have

$$\nabla_a n^a_{\rm x} = 0$$

and the variational approach leads to

$$2n^a_{\rm x}\nabla_{[a}\bar{\mu}^{\rm x}_{b]} = 2j^a_{\rm x}\nabla_{[b}A_{a]} = j^a_{\rm x}F_{ba}$$

where

$$\bar{\mu}^{\rm x}_a = \mu^{\rm x}_a + q^{\rm x}A_a$$

and the stress-energy tensor is

$$T_{ab} = \sum_{\rm x} n^{\rm x}_a\bar{\mu}^{\rm x}_a + g_{ab}\left(\Lambda - \sum_{\rm x} n^a_{\rm x}\bar{\mu}^{\rm x}_c\right) + T^{\rm EM}_{ab}$$

where does the current go?

Also find the "electromagnetic" contribution to the stress-energy tensor;

$$T_{ab}^{\mathrm{EM}} = \frac{1}{\mu_0}\left[g^{cd}F_{ac}F_{bd} - \frac{1}{4}g_{ab}\left(F_{cd}F^{cd}\right)\right]$$

Note that the current does not enter this expression. Consider the coupling term;

$$I_{\mathrm{C}} = \int j^a A_a \sqrt{-g}\ d^4x$$

leading to

$$\delta\left(n^a A_a \sqrt{-g}\right) = \sqrt{-g}\left[A_a \delta n^a + n^a \delta A_a\right] + n^a A_a \delta\sqrt{-g}$$

$$= \sqrt{-g}\left(2\xi^a n^b \nabla_{[a}A_{b]} + n^a \delta A_a\right)$$

This illustrates the importance of the constraint in the variational principle.

fibration

Consider a two-fluid systems (p and e), with each kind of particle carrying a single unit of charge. Decompose each flux using

$$u_{\mathrm{x}}^a = \gamma_{\mathrm{x}} \left(u^a + v_{\mathrm{x}}^a \right) \qquad u^a v_a^{\mathrm{x}} = 0 \qquad \gamma_{\mathrm{x}} = \left(1 - v_{\mathrm{x}}^2 \right)^{-1/2}$$

then define v^a and w^a, such that

$$\left(P + \rho \right) v^a = n_{\mathrm{p}} \mu_{\mathrm{p}} v_{\mathrm{p}}^a + n_{\mathrm{e}} \mu_{\mathrm{e}} v_{\mathrm{e}}^a$$

$$w^a = v_{\mathrm{p}}^a - v_{\mathrm{e}}^a$$

This means that the current can be decomposed as

$$j^a = e \left(n_{\mathrm{p}} - n_{\mathrm{e}} \right) \left(u^a + v^a \right) + e \frac{n_{\mathrm{p}} n_{\mathrm{e}}}{P + \rho} \left(\mu_{\mathrm{p}} + \mu_{\mathrm{e}} \right) w^a$$

Choose observer such that v^a=0, and work only to linear order in w^a to simplify the problem. (Relativistic analogue of centre-of-mass frame.) Then;

$$J^a = e n_{\mathrm{e}} w^a$$

resistivity

Add (phenomenological) resistivity, representing linear scattering;

$$\tilde{f}^a_{\rm p} = e\mathcal{R} \perp^{ab}_{\rm p} n^{\rm e}_b = -\mathcal{R} \perp^{ab}_{\rm p} j_b$$

$$\tilde{f}^a_{\rm e} = e\mathcal{R} \perp^{ab}_{\rm e} n^{\rm p}_b = \mathcal{R} \perp^{ab}_{\rm e} j_b$$

Note: At linear level we have

$$\tilde{f}^a_{\rm p} + \tilde{f}^a_{\rm e} = e\mathcal{R} \left(\perp^{ab}_{\rm p} n^{\rm e}_b + \perp^{ab}_{\rm e} n^{\rm p}_b \right) \approx e\mathcal{R} \left(n_{\rm p} - n_{\rm e} \right) w^a$$

Need charge neutral system in order to retain energy conservation.

This is as expected. In general, the resistivity will generate heat, which is not accounted for in the two-component model. To do this, we need to add an entropy component and consider the associated heat flux (as well as the second law etcetera).

Ohm's law

Working out the (suitably!) weighted difference between the two momentum equations we arrive at the relativistic form for Ohm's law.

The general result is quite complicated, but there are two instructive examples.

For pair plasmas we have

$$E_b - \frac{\mathcal{R}}{en_\mathrm{e}} J_b = \frac{\mu}{2e^2 n_\mathrm{e}} \left[\perp_{ab} \dot{j}^a + J^a \left(\sigma_{ab} + \omega_{ab} + \frac{4}{3}\theta \perp_{ab} \right) \right]$$

while proton-electron plasmas are described by

$$E_b - \frac{1}{en_\mathrm{e}} \epsilon_{bcd} J^c B^d - \frac{\mathcal{R}}{en_\mathrm{e}} J_b$$
$$= \frac{\mu_\mathrm{e}}{e^2 n_\mathrm{e}} \left[\perp_{ab} \dot{j}^a + J^a \left(\sigma_{ab} + \omega_{ab} + \frac{4}{3}\theta \perp_{ab} \right) \right] - \frac{1}{e} \perp^a{}_b \nabla_a \mu_\mathrm{e}$$

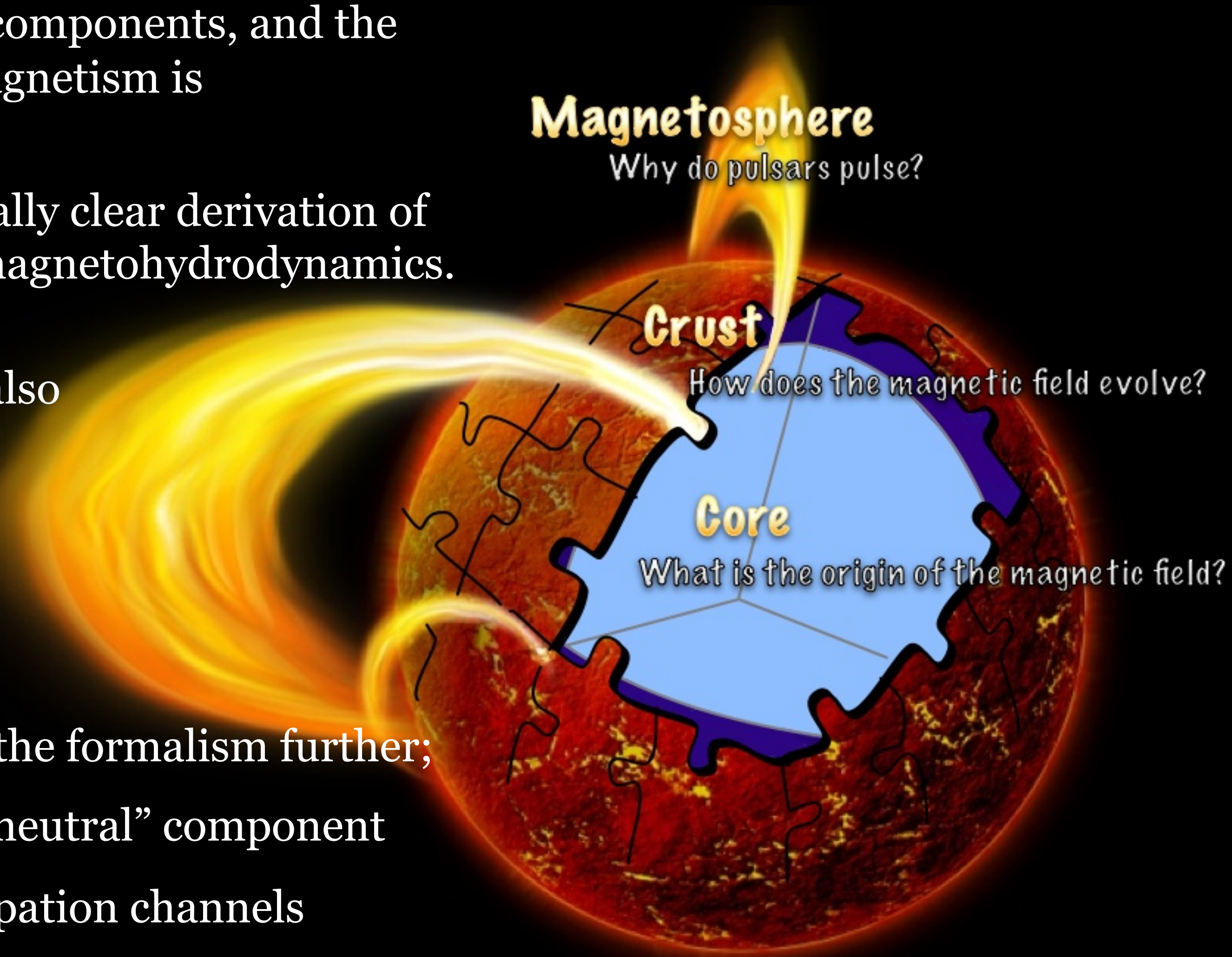
The absence/presence of i) the Hall term and ii) the battery term is notable.

Various approximations follow by neglecting terms in these expressions. Ideal MHD is the most "extreme" – more an assumption than an approximation!

remarks

The variational multi-fluid framework is easily extended to charged components, and the coupling to electromagnetism is straightforward.

Provides as conceptually clear derivation of resistive relativistic magnetohydrodynamics.



The model can be extended to account also for thermal effects (entropy) and may be applied to a range of relevant astrophysics problems.

Also need to develop the formalism further;

- the presence of a “neutral” component
- more general dissipation channels
- superconductivity