A Newtonian problem as an insightful tool for the behavior of gravitational-wave sources

Theocharis Apostolatos

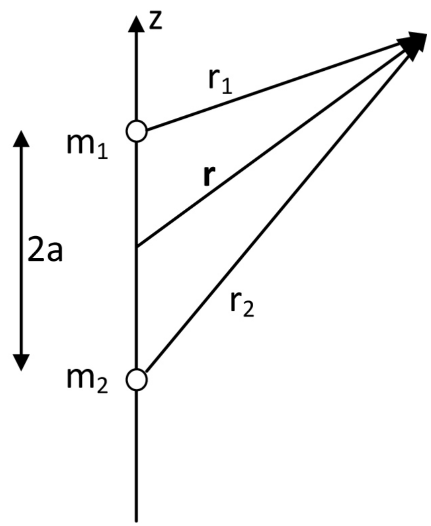
University of Athens

Chania, June 20, 2012

The Euler's problem

- The gravitational field of two fixed point-masses (~ 1760):

$$
\begin{aligned}
V &= -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} \\
&= -\frac{Gm_1}{|\mathbf{r} - a\hat{\mathbf{z}}|} - \frac{Gm_2}{|\mathbf{r} + a\hat{\mathbf{z}}|}
\end{aligned}
$$



- Better analyzed in spheroidal coordinates:

$$
\xi = \frac{r_1 + r_2}{2a}, \eta = \frac{r_1 - r_2}{2a}
$$

1st similarity

- This Newtonian problem is integrable, just like Kerr. That is, there are as many integrals of motion as the number of dimensions.

$$E\ ,\ L_z\ , b$$

- Integrability renders the (geodesic) motion directly solvable through the Hamilton-Jacobi method.

2nd similarity

- The gravitational field of a Kerr b.h. is oblate, while Euler's potential is prolate. By replacing $a \to ia$, Euler's problem becomes oblate as well.

$$\begin{aligned} V &= \frac{M}{\sqrt{2}R^2}\sqrt{R^2 + r^2 - a^2} = -\frac{M\xi}{a(\xi^2+\eta^2)}, \\ \text{where } R^2 &= \sqrt{(r^2-a^2)^2 + (2a\mathbf{r}\cdot\hat{\mathbf{z}})^2}. \end{aligned}$$

- As in Kerr, closed orbits precess around the equatorial plane, due to its non-vanishing quadrupole moment. No frame dragging in the oblate version of Euler's problem though.

3rd similarity

- The Carter constant for the Kerr metric is ($\mu_{\text{test body}} = 1$)

$$Q = u_\theta^2 + \cos^2\theta \left[a^2(1 - E^2) + L_z^2/\sin^2\theta\right]$$

- The 3rd integral of motion in the Euler problem is

$$b = -Q - L_z^2 - 2a^2 E_{\rm N}$$

if we make the following (rational) replacements

$$a \to ia\ ,\ \eta \to \cos\theta\ ,\ p_\eta^2(1-\eta^2) \to p_\theta^2\ ,\ E^2 \to 1 + 2E_{\rm N}$$

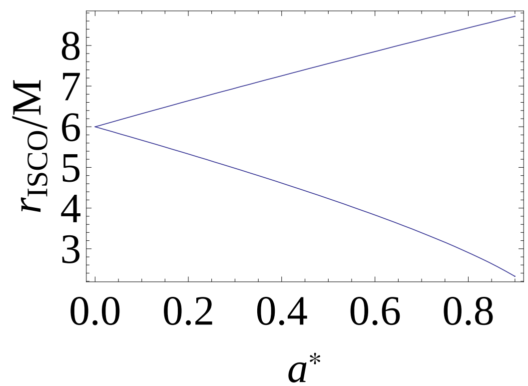
Thus the physical meaning of the new integral of motion is similar.

4th similarity (in collaboration with K. Chatziioannou)

- The innermost stable circular orbit (ISCO) as a function of a is

$$\frac{r_{\rm ISCO}}{M} = 3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}$$

$$Z_1 = 1 + \sqrt[3]{1 - a^{\star 2}} \left(\sqrt[3]{1 - a^\star} + \sqrt[3]{1 + a^\star} \right) , \; Z_2 = \sqrt{3a^{\star 2} + Z_1^2} , \; a^\star = a/M.$$



- For the (the oblate version of the) Euler problem there is also an ISCO

$$\frac{\xi_{\rm ISCO}}{a} = \sqrt{3}.$$

5th similarity (in collaboration with K. Chatziioannou)

- The invariant frequencies related to circular orbits, perihelion precession, and azimuthal precession in Kerr are the ones that could be directly measured (through e.g. gravitational waves).

- Since the Euler problem is integrable as well, one can calculate the corresponding frequencies. Although the frequencies are not expressed by the same functions they share common qualitative characteristics: [(i) $\omega_r/\omega_\theta \to 1$ for $r \to \infty$, (ii) the resonance condition $\omega_r/\omega_\theta = 2$ is never met, (iii) for a range of parameters (a, ι =orbital inclination) the ratio of frequencies have similar orbital-semilatus-rectum-dependence.]

...consequence of no resonance (in collaboration with K. Chatziioannou)

- The non-resonant condition $\omega_r/\omega_\theta \neq 2n$ ensures that an initial circular ($r =$ const) orbit will evolve adiabatically to a circular one, due to radiation reaction in Kerr (Ori, Kenefick (1995)). Exactly the same non-resonant condition in Euler problem ensures adiabatic "circularity" of orbits. Actually the Newtonian case is easier to explore and understand in a deeper way the arguments of Ori and Kenefick.

6th similarity (Will (2010))

- The relativistic (Geroch-Hansen) multipole moments of Kerr are

$$M_{2k} = M(-a^2)^k \ , \ S_{2k+1} = Ma(-a^2)^k$$

- Will showed that in Newtonian gravity the only axially symmetric, and with reflection symmetry potential that posesses a Carter-like constant is the Euler potential which has exactly the same spectrum of mass moments:

$$\begin{aligned} M_{2k} &= M(a^2)^k \ \text{(prolate)} \\ M_{2k} &= M(-a^2)^k \ \text{(oblate)} \end{aligned}$$

7th similarity (in collaboration with K. Glampedakis)

- Kerr metric provides a separable set of differential equations when analysing the propagation of perturbations in the corresponding background.

- Looking for a Newtonian axially-symmetric potential that renders the wave-equation
$$\nabla^2\Psi + (\omega^2/c^2 - \kappa V)\Psi = 0$$
separable, one obtains two such solutions:
  i. Working with spherical coordinates: the monopole solution.
  ii. Working with spheroidal coordinates: the Euler problem.

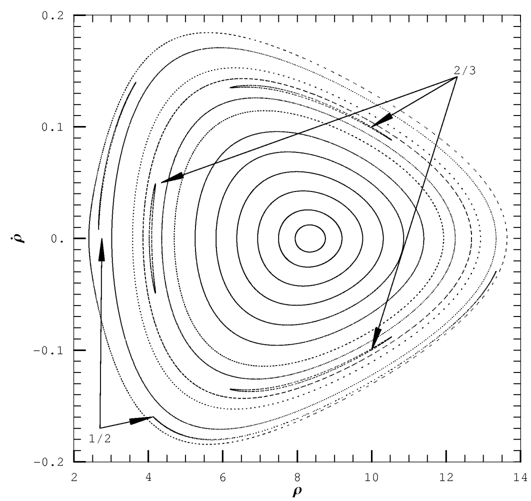
- Moreover for $\kappa = 4\omega^2$ the corresponding pair of azimuthal and radial differential equations are (1) identical with respect to the azimuthal o.d.e. and (2) extremely similar with respect to the radial part, compared with the corresponding Kerr equations.

8th similarity (in collaboration with G. Pappas)

- A Poincaré section (a stroboscopic picture of the phase space) of orbits in Kerr is a regular pattern of closed curves nested within each other (integrability).

- The same pattern of very similar series of nested closed curves arises in the Euler problem due to its integrable nature.

- When a Kerr metric is perturbed (slightly non-integrable), the tori of regular orbits slightly deform, but they still exist. The tori that correspond to resonances ($\omega_r/\omega_\theta = p/q$ (rational)) disintegrate and a Birkhoff chain of islands form in the Poincaré section instead of a single curve.



- Exactly the same picture forms when analysing a perturbed Euler problem (e.g. by putting a small mass at the origin).

Gaining insight from the Newtonian problem (in collaboration with G. Pappas)

- The analysis of resonances in the slightly non-Kerr metric offers an opportunity to look for non-Kerr (alternative gravity theories) objects through gravitational wave analysis.

- When the orbit evolves through a resonance there are qualitative new features that show up (ratios of precessing frequencies remain fixed). However, quantitatively analysing such a problem in the framework of relativity is not an easy task (self-force is not known).

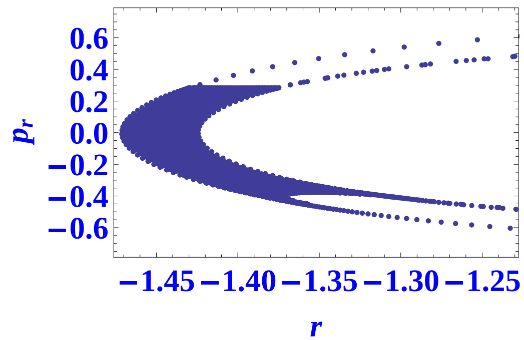
- The slightly perturbed Euler problem is an excellent paradigm to study such characteristics. We have tried evolving orbits in the perturbed Euler problem by implementing a phenomelogical friction that mimics the effect of gravitational radiation

$$\mathbf{F}_{\mathrm{GR}} \propto -F_{\mathrm{G}} \left(\frac{u}{c}\right)^4 \frac{\mathbf{u}}{c}$$

Gaining insight … (in collaboration with G. Pappas)

- Using a self-force in the Newtonian problem we could check if our understanding of evolving an orbit near a resonance, that is based in average energy and angular-momentum losses, is right or not.

- First numerical results suggests that the self-force description leads to a much longer time to cross a resonance, compared to what one gets by assuming a constant average loss: $\left\langle \dot{E} \right\rangle$, and $\left\langle \dot{L}_z \right\rangle$.



END