Stability and Instability of Extremal Black Holes

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Recent Developments in Gravity, Chania, Crete

Introduction

- *Einstein's general relativity theory*: The study of (4-dimensional) Lorentzian manifolds $(\mathcal{M}, g)$ which satisfy the Einstein equations:

$$Ric - \frac{1}{2} g \cdot R_{\mathsf{sc}} = \mathbf{T}.$$

- *Black holes* are one of most celebrated predictions of the theory, namely the existence of spacetimes for which we have a well-defined *complete* infinity $\mathcal{I}$ where radiation escapes and such that

$$\mathcal{M} - J^{-}(\mathcal{I}) \neq \emptyset.$$

  Then, $\mathcal{M} - J^{-}(\mathcal{I})$ is the black hole region, $J^{-}(\mathcal{I})$ is the *domain of outer communications* and $\partial J^{-}(\mathcal{I})$ is the *event horizon* (denoted by $\mathcal{H}$).

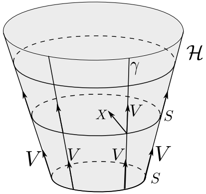
- Under appropriate causality conditions (e.g. global hyperbolicity), the event horizon $\mathcal{H}$ is a *null hypersurface*, $\mathcal{H}$ being the boundary of the past of a set.

Null Hypersurfaces and the Surface Gravity κ

- Null hypersurface $H \subset \mathcal{M}$: $\forall p \in H$ the tangent plane $T_pH = \langle V \rangle^{\perp}$ and V is null (hence T_pH is degenerate). Then, $X \in T_pH$ is null if $X \in \langle V \rangle$ and spacelike otherwise.
- $g(\nabla_V V, X) = -g(V, \nabla_V X) = -g(V, \nabla_X V) - g(V, [V, X]) = -\frac{1}{2} X\big(g(V, V)\big) = 0$, hence

$$\nabla_V V = \kappa V$$

  and so the integral curves γ of V are geodesics (κ: surface gravity).



- Killing horizon H: V Killing and so κ is constant along γ.
- Z.L.o.B.H.M.: If $(\mathcal{M}, g)$ satisfies $Ric(g) = 0$ (and V is Killing) then κ is globally constant on H.
- Extremal horizon: Killing horizon with $\kappa = 0$ (subextremal: $\kappa > 0$). Null generators are affinely parametrized. No bifurcate sphere.

The Main Examples

- Extremal Kerr
- Extremal Reissner–Nordström
- Majumda–Papapetrou multi black holes

Importance of Extremal Black Holes/Known Results

- Classical Physics: No redshift effect along the event horizon $\mathcal{H}$.
- Quantum Physics: Zero temperature and hence extremal black holes do not radiate.
- Geometry/Analysis:
  - If $\underline{\chi}$ is the transversal second fundamental form of the sections of a vacuum extremal horizon $\mathcal{H}$, then $\mathcal{L}_V \underline{\chi} = 0$.
    1. For extremal vacuum horizons the torsion η satisfies an elliptic system.
    2. No static vacuum extremal horizons with spherical topology (Chruściel, Reall, Tod).
    3. Rigidity of geometry of (electro-)vacuum axisymmetric extremal horizons: $\not{g}, \eta, \rho, \sigma$ are fully determined (after fixing a gauge) (Hájíček, Lewandowski and Pawlowski, Kunduri and Lucietti).
  - Static electrovacuum spacetime with many black holes $\Rightarrow$ all black holes are extremal (Chruściel, Tod). Example: Majumdar–Papapetrou.

The Wave Equation

- We initiate the study of the wave equation

$$\Box_g \psi = 0$$

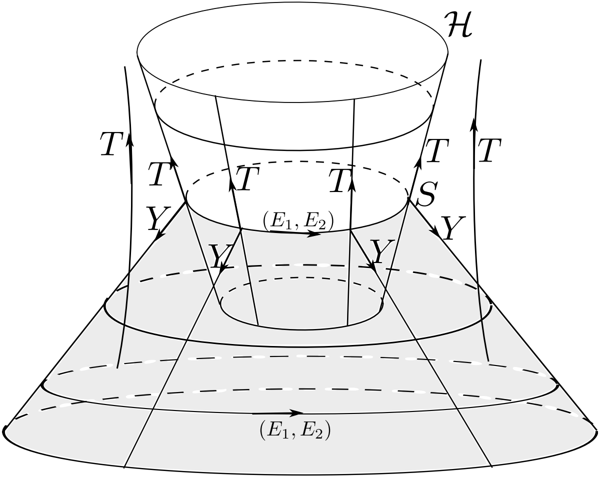
  in the exterior region of extremal black holes up to and including the event horizon.
- No previous (mathematical, numerical or heuristic) results known for asymptotics of waves along extremal horizons.
- We start by considering extremal Reissner–Nordström backgrounds.

New features of Extremal Horizons

- There exists a conservation law along the event horizon for the spherical mean.
- This law had not been previously observed.
- For the case of extremal Reissner–Nordström, the proof of this law is relatively simple and so we will present essentially all the details.
- We first introduce a frame that will be very useful for our analysis.

Local Geometry of Extremal Reissner–Nordström

The T-propagated frame (T, Y, E_1, E_2):
(If r is the radius of the spheres of symmetry, then $Y = \partial_r$.)



A Conservation Law for Extremal Reissner–Nordström

Let $M > 0$ denote the mass. Then $\mathcal{H} = \{r = M\}$. If we write the wave equation using the (T, Y, E_1, E_2) frame we obtain

$$D \cdot (YY\psi) + 2(TY\psi) + \frac{2}{r} \cdot (T\psi) + \left(D' + \frac{2D}{r}\right) \cdot (Y\psi) + \not{\Delta}\psi = 0,$$

where

$$D = g(T,T) = \left(1 - \frac{M}{r}\right)^2.$$

Assume $\not{\Delta}\psi = 0$. Then, since $D = D' = 0$ on the horizon $\mathcal{H}$, we have

$$T\left(Y\psi + \frac{1}{M}\psi\right) = 0$$

and since T is tangential to $\mathcal{H}$, the quantity

$$H[\psi] = Y\psi + \frac{1}{M}\psi$$

is conserved along the event horizon $\mathcal{H}$ for all spherically symmetric solutions ψ.

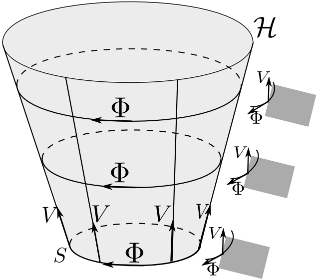
Generalisations?

- What about, for example, extremal Kerr or Majumdar–Papapetrou spacetime?

Generalised Conservation Law

Theorem (S.A.)

Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold containing an extremal axisymmetric horizon $\mathcal{H}$.
Let also V denote the Killing field null and normal to $\mathcal{H}$ and Φ denote the axial Killing Φ tangential to $\mathcal{H}$ and such that $[V, \Phi] = 0$. If the distribution of the planes orthogonal to the planes spanned by V and Φ is integrable, then we have a conservation law on the horizon $\mathcal{H}$.



Applications

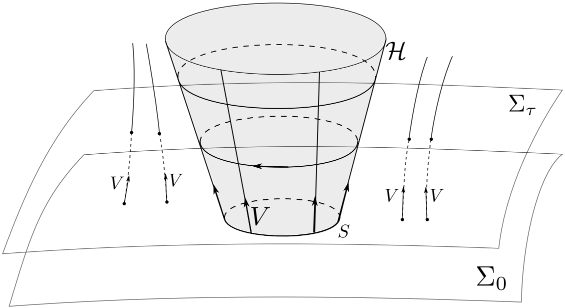
- The conservation law holds for the spherical mean of an expression of ψ and first order derivatives of ψ.
- Theorem holds for extremal Kerr. Explicitly, the quantity

$$H^{\mathsf{Kerr}}[\psi](\tau) = \int_{S_\tau} \left(M \sin^2\theta \,(T\psi) + 4M\,(Y\psi) + 2\psi \right)$$

  is conserved along the event horizon $\mathcal{H}$.
- Theorem holds for Majumdar–Papapetrou multi black holes.

Initial Value Problem

- The conservation laws are completely determined by the local properties of extremal horizons (namely, by the induced metric $\not{g}$ and the Christoffel symbols Γ on $\mathcal{H}$) and hence do not depend on global aspects of the spacetime.
- Hence we have not discussed global hyperbolicity or well-posedeness of the wave equation or other properties (behaviour of the geodesic flow etc.).
- Initial value problem for extremal Reissner–Nordström and extremal Kerr.



Instability Results

Theorem (S.A.)

For generic solutions ψ to the wave equation on extremal Reissner–Nordström or extremal Kerr backgrounds we have:

Non-Decay:

The translation-invariant transversal to $\mathcal{H}$ derivative $Y\psi$ does not decay along $\mathcal{H}$.

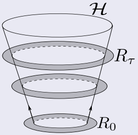
Pointwise Blow-up:

$$\left|Y^k\psi\right| \to +\infty,$$

along $\mathcal{H}$ as advanced time tends to infinity $k \geq 2$.

Energy Blow-up:

$$\left\|Y^k\psi\right\|_{L^2(R_\tau)} \to +\infty$$



as $\tau \to +\infty$ for all $k \geq 2$.

This result is in stark contrast with the subextremal case.

Stability Results

Theorem (S.A.)

For all solutions ψ (with sufficiently regular initial data on Σ_0) to the wave equation on extremal Reissner–Nordström and all axisymmetric solutions on extremal Kerr we have

$$\textbf{Pointwise Decay:} \ |\psi(\tau, \cdot)| \to 0$$

as $\tau \to +\infty$ up to and including the event horizon $\mathcal{H}$.

THANK YOU!