Charging axisymmetric space-times with cosmological constant

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Complex potential formalism   Equations of motion

$$I[g_{\mu\nu}, A_\mu] = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\frac{1}{G}\left(R-2\Lambda\right)-\frac{1}{\mu_0}F_{\mu\nu}F^{\mu\nu}\right]$$

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu} = 2\frac{G}{\mu_0}\left(F_{\mu\rho}F_\nu{}^\rho-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right)$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0 \qquad .$$

$$ds^2 = -\alpha e^{\Omega/2}\left(dt+\omega d\varphi\right)^2+\alpha e^{-\Omega/2}d\varphi^2+\frac{e^{2\nu}}{\sqrt{\alpha}}\left(\mathrm{d}\mathrm{r}^2+d\mathrm{z}^2\right)$$

$$EE^{\mathrm{r}}{}_{\mathrm{r}}+EE^{\mathrm{z}}{}_{\mathrm{z}} \quad : \quad \nabla^2\alpha+2\Lambda\sqrt{\alpha}e^{2\nu}=0$$

$$EE^{\varphi}{}_{t} \quad : \quad \frac{1}{2}\overrightarrow{\nabla}\cdot\left(\mathrm{e}^{\Omega}\alpha\overrightarrow{\nabla}\omega\right)+2\frac{G}{\mu_0}e^{\Omega/2}\left[\omega(\overrightarrow{\nabla}A_0)^2-\overrightarrow{\nabla}A_0\cdot\overrightarrow{\nabla}A_3\right]=0$$

$$EE^{t}{}_{t}-EE^{\varphi}{}_{\varphi} \quad : \quad \frac{1}{2}\overrightarrow{\nabla}\cdot\left(\alpha\overrightarrow{\nabla}\Omega\right)+\omega\overrightarrow{\nabla}\cdot\left(\mathrm{e}^{\Omega}\alpha\overrightarrow{\nabla}\omega\right)+\alpha e^{\Omega}(\overrightarrow{\nabla}\omega)^2+$$

$$+2\frac{G}{\mu_0}\left\{e^{\Omega/2}\left[(\overrightarrow{\nabla}A_0)^2\omega^2-(\overrightarrow{\nabla}A_3)^2\right]-e^{-\Omega/2}(\overrightarrow{\nabla}A_0)^2\right\}=0$$

$$EE^{t}{}_{t}+EE^{\varphi}{}_{\varphi} \quad : \quad \frac{1}{2}\alpha e^{\Omega}(\overrightarrow{\nabla}\omega)^2-\frac{1}{8}\alpha(\overrightarrow{\nabla}\Omega)^2+\frac{1}{2}\nabla^2\alpha-2\alpha\nabla^2\nu=0$$

$$EM^t \quad : \quad \overrightarrow{\nabla}\cdot\left[e^{-\Omega/2}\overrightarrow{\nabla}A_0+\omega e^{\Omega/2}\left(\overrightarrow{\nabla}A_3-\omega\overrightarrow{\nabla}A_0\right)\right]=0$$

$$EM^\varphi \quad : \quad \overrightarrow{\nabla}\cdot\left[e^{\Omega/2}\left(\overrightarrow{\nabla}A_3-\omega\overrightarrow{\nabla}A_0\right)\right]=0 \quad .$$

Complex potential formalism   Equations of motion

$$\vec{e}_\varphi \times \vec{\nabla}\tilde{A}_3 := e^{\Omega/2}\left(\vec{\nabla}A_3 - \omega\vec{\nabla}A_0\right)$$

$$\nabla \cdot \left(\vec{e}_\varphi \times \vec{\nabla}\tilde{A}_3\right) = 0 \qquad \Longrightarrow \qquad \partial_{\rm r}\partial_{\rm z}\tilde{A}_3 - \partial_{\rm z}\partial_{\rm r}\tilde{A}_3 = 0$$

$$\vec{e}_\varphi \times \vec{\nabla}A_3 = -\left(e^{-\Omega/2}\vec{\nabla}\tilde{A}_3 - \omega\vec{e}_\varphi \times \vec{\nabla}A_0\right)$$

$$EM^\varphi \quad : \quad \vec{\nabla}\cdot\left(e^{-\Omega/2}\vec{\nabla}\tilde{A}_3 - \omega\vec{e}_\varphi \times \vec{\nabla}A_0\right) = 0$$

$$EM^t \quad : \qquad \vec{\nabla}\cdot\left[e^{-\Omega/2}\vec{\nabla}A_0 + \omega e^{\Omega/2}\left(\vec{\nabla}A_3 - \omega\vec{\nabla}A_0\right)\right] = 0$$

$$\vec{\nabla}\cdot\left(e^{-\Omega/2}\vec{\nabla}A_0 + \omega\vec{e}_\varphi \times \vec{\nabla}\tilde{A}_3\right) = 0$$

$$\mathbf{\Phi} := A_0 + i\,\tilde{A}_3$$

$$\vec{\nabla}\cdot\left(e^{-\Omega/2}\vec{\nabla}\mathbf{\Phi} - i\,\omega\vec{e}_\varphi \times \vec{\nabla}\mathbf{\Phi}\right) = 0$$

Complex potential formalism Equations of motion

$$EE^{\varphi}{}_{t} \quad : \qquad \frac{1}{2}\overrightarrow{\nabla}\cdot\left(\mathrm{e}^{\Omega}\alpha\overrightarrow{\nabla}\omega\right)+2\frac{G}{\mu_0}e^{\Omega/2}\left[\omega(\overrightarrow{\nabla}A_0)^2-\overrightarrow{\nabla}A_0\cdot\overrightarrow{\nabla}A_3\right]=0$$

$$\overrightarrow{\nabla}\cdot\left[e^{\Omega}\alpha\overrightarrow{\nabla}\omega-2\overrightarrow{e}_{\varphi}\times\mathrm{Im}(\mathbf{\Phi}^*\overrightarrow{\nabla}\mathbf{\Phi})\right]=0$$

$$\overrightarrow{e}_{\varphi}\times\overrightarrow{\nabla}h:=e^{\Omega}\alpha\overrightarrow{\nabla}\omega-2\overrightarrow{e}_{\varphi}\times\mathrm{Im}(\mathbf{\Phi}^*\overrightarrow{\nabla}\mathbf{\Phi})$$

$$\overrightarrow{\nabla}\cdot\left\{e^{-\Omega}\alpha^{-1}\left[\overrightarrow{\nabla}h+2\ \mathrm{Im}(\mathbf{\Phi}^*\overrightarrow{\nabla}\mathbf{\Phi})\right]\right\}=0$$

$$EE^{t}{}_{t}-EE^{\varphi}{}_{\varphi} \qquad \frac{f}{\alpha}\overrightarrow{\nabla}\cdot\left(\alpha\overrightarrow{\nabla}f\right)-\overrightarrow{\nabla}f\cdot\overrightarrow{\nabla}f-f^2\frac{\nabla^2\alpha}{\alpha}=2f\overrightarrow{\nabla}\mathbf{\Phi}\cdot\overrightarrow{\nabla}\mathbf{\Phi}^*-\left[\overrightarrow{\nabla}h+2\ \mathrm{Im}(\mathbf{\Phi}^*\overrightarrow{\nabla}\mathbf{\Phi})\right]^2$$

$$f:=e^{\Omega/2}\alpha. \qquad\qquad \mathcal{E}:=f-|\mathbf{\Phi}|^2+i\ h$$

$$\begin{aligned}
\left(\mathrm{Re}\ \mathcal{E}+|\mathbf{\Phi}|^2\right)\frac{1}{\alpha}\overrightarrow{\nabla}\cdot\left(\alpha\overrightarrow{\nabla}\mathcal{E}\right) &= \left(\overrightarrow{\nabla}\mathcal{E}+2\ \mathbf{\Phi}^*\overrightarrow{\nabla}\mathbf{\Phi}\right)\cdot\overrightarrow{\nabla}\mathcal{E}+\mathrm{Re}^2\left(\mathcal{E}+|\mathbf{\Phi}|^2\right)\frac{\nabla^2\alpha}{\alpha}\\
\left(\mathrm{Re}\ \mathcal{E}+|\mathbf{\Phi}|^2\right)\frac{1}{\alpha}\overrightarrow{\nabla}\cdot\left(\alpha\overrightarrow{\nabla}\mathbf{\Phi}\right) &= \left(\overrightarrow{\nabla}\mathcal{E}+2\ \mathbf{\Phi}^*\overrightarrow{\nabla}\mathbf{\Phi}\right)\cdot\overrightarrow{\nabla}\mathbf{\Phi}\quad .
\end{aligned}$$

Complex potential formalism   Effective action and symmetries

$$S = \int \mathrm{d}r\mathrm{d}z\ \alpha \left[\frac{(\vec{\nabla}\mathcal{E} + 2\ \mathbf{\Phi}^*\vec{\nabla}\mathbf{\Phi})\cdot(\vec{\nabla}\mathcal{E}^* + 2\ \mathbf{\Phi}\vec{\nabla}\mathbf{\Phi}^*)}{(\mathcal{E}+\mathcal{E}^*+2\mathbf{\Phi}\mathbf{\Phi}^*)^2} - \frac{2\vec{\nabla}\mathbf{\Phi}\cdot\vec{\nabla}\mathbf{\Phi}^* + \frac{\vec{\nabla}\alpha}{2\alpha}\cdot\vec{\nabla}(\mathcal{E}+\mathcal{E}^*+2\mathbf{\Phi}\mathbf{\Phi}^*)}{\mathcal{E}+\mathcal{E}^*+2\mathbf{\Phi}\mathbf{\Phi}^*} \right]$$

in the vacuum case, that is for $\mathbf{\Phi} = 0$: $S = \int \mathrm{d}r\mathrm{d}z\ \alpha \left\{ \frac{\vec{\nabla}\mathcal{E}\cdot\vec{\nabla}\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} - \frac{\vec{\nabla}\alpha}{2\alpha}\cdot\frac{\vec{\nabla}\mathcal{E}+\vec{\nabla}\mathcal{E}^*}{\mathcal{E}+\mathcal{E}^*} \right\}$

$$\mathcal{E} \longrightarrow \mathcal{E}' = \frac{a\mathcal{E}+ib}{ic\mathcal{E}+d} \quad , \qquad \text{where} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

$i)$ $\mathcal{E} \longrightarrow \mathcal{E}' = \lambda^2\mathcal{E}$, $(a=\lambda,\ b=0,\ c=0,\ d=1/\lambda)$

$ii)$ $\mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} + i\ b$, $(a=1,\ b=b,\ c=0,\ d=1)$

$iii)$ $\mathcal{E} \longrightarrow \mathcal{E}' = \frac{\mathcal{E}}{1+ic\mathcal{E}}$, $(a=1,\ b=0,\ c=c,\ d=1)$

$$\chi = (\lambda x + cxy)\partial_x + [b + \lambda y + c(x^2 + y^2)]\partial_y$$

Complex potential formalism   Effective action and symmetries

$$S = \int d\mathrm{r} d\mathrm{z}\, \alpha \left[\frac{(\vec{\nabla}\mathcal{E} + 2\, \mathbf{\Phi}^* \vec{\nabla}\mathbf{\Phi}) \cdot (\vec{\nabla}\mathcal{E}^* + 2\, \mathbf{\Phi} \vec{\nabla}\mathbf{\Phi}^*)}{(\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*)^2} - \frac{2\vec{\nabla}\mathbf{\Phi} \cdot \vec{\nabla}\mathbf{\Phi}^* + \frac{\vec{\nabla}\alpha}{\alpha} \cdot \vec{\nabla}(\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*)}{\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*} \right]$$

SU(2,1):

$$I) \qquad \mathcal{E} \longrightarrow \mathcal{E}' = \lambda\lambda^*\mathcal{E} \quad , \quad \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \lambda\lambda^*\mathbf{\Phi}$$

$$II) \qquad \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} + i\, b \quad , \quad \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \mathbf{\Phi}$$

$$III) \qquad \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E}/(1 + ic\mathcal{E}) \quad , \quad \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \mathbf{\Phi}/(1 + ic\mathcal{E})$$

$$IV) \qquad \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} - 2\beta^*\mathbf{\Phi} - \beta\beta^* \quad , \quad \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \mathbf{\Phi} + \beta$$

$$V) \qquad \mathcal{E} \longrightarrow \mathcal{E}' = \frac{\mathcal{E}}{1 - 2\gamma^*\mathbf{\Phi} - \gamma\gamma^*\mathcal{E}} \quad , \quad \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \frac{\mathbf{\Phi} + \gamma\mathcal{E}}{1 - 2\gamma^*\mathbf{\Phi} - \gamma\gamma^*\mathcal{E}}$$

Example: Electrifying the Kerr-(A)dS Black Hole

Kerr-(A)dS metric

$$ds^2 = -\frac{\Delta_r}{\varrho^2}\left(dt - a\ \sin^2\theta\ d\varphi\right)^2 + \varrho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta\ \sin^2\theta}{\varrho^2}\left[adt - (r^2 + a^2)d\varphi\right]^2$$

$$\begin{aligned} \varrho^2 &= r^2 + a^2\cos^2\theta \\ \Delta_\theta &= 1 + \frac{\Lambda}{3}a^2\cos^2\theta \\ \Delta_r &= (r^2 + a^2)\left(1 - \frac{\Lambda}{3}r^2\right) - 2Mr \end{aligned}$$

$$\begin{aligned} \alpha &= \sin\theta\sqrt{\Delta_r\Delta_\theta} \\ e^{\Omega/2} &= \frac{\Delta_r - a^2\Delta_\theta\sin^2\theta}{\sqrt{\Delta_r\Delta_\theta}\varrho^2\sin\theta} \\ \omega &= \frac{a\sin^2\theta\left[\Delta_r - \Delta_\theta(r^2 + a^2)\right]}{a^2\Delta_\theta\sin^2 - \Delta_r} \end{aligned}$$

Ernst complex potential $\qquad \mathcal{E} = \alpha e^{\Omega/2} + ih = \frac{\xi - 1}{\xi + 1} + \frac{1}{\ell^2}\left[(\xi + 1)^2 + q^2\right]$

$$\xi = px - iqy \quad , \quad p = \frac{k}{M} \quad , \quad x = \frac{r - M}{k} \quad , \quad q = \frac{a}{M} \quad , \quad y = \cos\theta \quad , \quad k = \sqrt{M^2 - a^2}$$

Charging Kerr-(A)dS metric

$$\frac{d^2\mathcal{E}}{d\mathbf{\Phi}^2}\ (\overrightarrow{\nabla}\mathbf{\Phi})^2\ \overrightarrow{\nabla}\mathbf{\Phi} = \nabla^2\mathcal{E}\ \overrightarrow{\nabla}\mathbf{\Phi} - \nabla^2\mathbf{\Phi}\ \overrightarrow{\nabla}\mathcal{E}$$

$$\frac{d^2\mathcal{E}}{d\mathbf{\Phi}^2}(\overrightarrow{\nabla}\mathbf{\Phi})^2 = \left(\mathrm{Re}\mathcal{E} + |\mathbf{\Phi}|^2\right)\frac{\nabla^2\alpha}{\alpha}$$

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_\ell, \qquad \frac{d^2\mathcal{E}_0}{d\mathbf{\Phi}^2} = 0 \qquad \Longrightarrow \qquad \mathcal{E}_0(\mathbf{\Phi}) = c_0 + c_1\ \mathbf{\Phi}$$

$$\mathcal{E}_0 = (\xi - 1)/(\xi + 1). \qquad c_1 := -2/Q \qquad Q := (e + ig)/M$$

$$\mathcal{E}_0 \to 1 \text{ and } \mathbf{\Phi} \to 0. \qquad \mathbf{\Phi} = \frac{Q}{\xi + 1}$$

$$\begin{aligned} f &= \mathrm{Re}(\mathcal{E} + |\mathbf{\Phi}|^2) = \frac{\xi^*\xi - 1 + |Q|^2}{|\xi + 1|^2} + \frac{1}{\ell^2}\,\mathrm{Re}\left[(\xi + 1)^2 + q^2\right] = \\ &= 1 - \frac{2Mr - e^2 - g^2}{r^2 + a^2\cos^2\theta} + \frac{1}{\ell^2 M^2}\left[r^2 + a^2\sin^2\theta\right] = \frac{\Delta_e - a^2\sin^2\theta\Delta_\theta}{\varrho^2}\quad, \\ h &= \mathrm{Im}(\mathcal{E}) = \frac{2\ \mathrm{Im}(\xi)}{|\xi + 1|^2} + \frac{1}{\ell^2}\,\mathrm{Im}\left(\xi^2 + 2\xi\right) = -2a\cos\theta\left(\frac{M}{r^2 + a^2} + \frac{r}{\ell^2 M^2}\right) \end{aligned}$$

$$\vec{e}_\varphi \times \vec{\nabla}\omega = -\alpha^{-1} e^{-\Omega}\left[\vec{\nabla} h + 2\ \mathrm{Im}(\mathbf{\Phi}^* \vec{\nabla}\mathbf{\Phi})\right]$$

$$\partial_z\omega\left[\partial_z h + 2\ \mathrm{Im}(\mathbf{\Phi}^*\partial_z\mathbf{\Phi}))\right] = -\partial_r\omega\left[\partial_r h + 2\ \mathrm{Im}(\mathbf{\Phi}^*\partial_r\mathbf{\Phi}))\right]$$
$$\alpha e^{\Omega} = \left[\partial_r h + 2\ \mathrm{Im}(\mathbf{\Phi}^*\partial_r\mathbf{\Phi})\right]/\partial_z\omega$$

$$\begin{aligned}
\alpha &= \sin\theta\sqrt{\Delta_e\Delta_\theta} \\
e^{\Omega/2} &= \frac{\Delta_e - a^2\Delta_\theta\sin^2\theta}{\sqrt{\Delta_e\Delta_\theta}\varrho^2\sin\theta} \\
\omega &= \frac{a\sin^2\theta\left[\Delta_e - \Delta_\theta(r^2+a^2)\right]}{a^2\Delta_\theta\sin^2 - \Delta_e} \\
e^{2\nu} &= \varrho^2\sqrt{\alpha} \qquad .
\end{aligned}$$

$$\Delta_e = \Delta_r + e^2 + g^2.$$

$$\mathbf{\Phi} := A_0 + i\ \tilde{A}_3 \qquad \vec{e}_\varphi \times \vec{\nabla}\tilde{A}_3 := e^{\Omega/2}\left(\vec{\nabla}A_3 - \omega\vec{\nabla}A_0\right)$$

$$A_\mu = \left(\ \frac{er - ga\cos\theta}{r^2 + a^2\cos^2\theta}\ ,\ 0\ ,\ 0\ ,\ \frac{-era\sin^2\theta + gy(r^2+a^2)}{r^2 + a^2\cos^2\theta}\ \right)$$

$$\mathrm{r} = \int \frac{dr}{\sqrt{r^2 + a^2 - \frac{\Lambda}{3}r^4 - \frac{\Lambda}{3}r^2a^2 - 2Mr + e^2 + g^2}} \quad , \qquad \mathrm{z} = \int \frac{d\theta}{1 + \frac{\Lambda}{3}a^2\cos^2\theta}$$

Magnetic universe with cosmological constant

$$ds^2 = \alpha e^{-\Omega/2} dt^2 - \alpha e^{\Omega/2} d\varphi^2 + \frac{e^{2\nu}}{\sqrt{-\alpha}} \left(d\mathrm{r}^2 + d\mathrm{z}^2\right)$$

$$ds^2 = \left(1 + \frac{\rho^2}{4}\right)^2 \left[-dt^2 + d\mathrm{z}^2 + \frac{\rho^2 \; d\rho^2}{k\left(1+\frac{\rho^2}{4}\right) - \frac{4\Lambda}{3}\left(1+\frac{\rho^2}{4}\right)^4}\right] + \frac{k\left(1+\frac{\rho^2}{4}\right) - \frac{4\Lambda}{3}\left(1+\frac{\rho^2}{4}\right)^4}{\left(1+\frac{\rho^2}{4}\right)^2} d\varphi^2$$

$$\mathcal{E}_0 = f_0 = \alpha_0 e^{\Omega_0/2} = -\frac{k}{1+\frac{\rho^2}{4}} + \frac{4\Lambda}{3}\left(1+\frac{\rho^2}{4}\right)^2$$

$$\mathbf{\Phi} = \frac{\pm 2B}{1+\rho^2/4} \qquad \Longrightarrow \qquad f = \mathrm{Re}(\mathcal{E} + |\mathbf{\Phi}|^2) = -\frac{k}{1+\frac{\rho^2}{4}} + \frac{4\Lambda}{3}\left(1+\frac{\rho^2}{4}\right)^2 + \frac{4B^2}{\left(1+\frac{\rho^2}{4}\right)^2}$$

$$\begin{aligned}
\alpha(\rho) &= -\sqrt{k\left(1+\frac{\rho^2}{4}\right) - 4B^2 - \frac{4\Lambda}{3}\left(1+\frac{\rho^2}{4}\right)^4} \\
e^{\Omega/2}(\rho) &= -\alpha(\rho)\left(1+\frac{\rho^2}{4}\right)^{-2} \\
e^{2\nu}(\rho) &= \sqrt{-\alpha(\rho)}\left(1+\frac{\rho^2}{4}\right)^2 \quad .
\end{aligned}$$

$$ds^2 = \left(1+\frac{\rho^2}{4}\right)^2\left[-dt^2+dz^2+\frac{\rho^2\,d\rho^2}{\alpha^2(\rho)}\right]+\frac{\alpha^2(\rho)}{\left(1+\frac{\rho^2}{4}\right)^2}d\varphi^2$$

$$\alpha(\rho) = -\sqrt{k\left(1+\frac{\rho^2}{4}\right)-4B^2-\frac{4\Lambda}{3}\left(1+\frac{\rho^2}{4}\right)^4}$$

$$k = 4B^2+4\Lambda/3, \quad \text{to avoid conical singularity around } \rho\approx 0$$

$$ds^2 = \left(1+\frac{B^2\bar\rho^2}{4}\right)^2\left[-dt^2+dz^2+\frac{d\bar\rho^2}{1-\frac{\Lambda}{3}\left(\frac{3}{B^2}+\frac{3\bar\rho^2}{2}+\frac{B^2\bar\rho^4}{4}+\frac{B^4\bar\rho^6}{64}\right)}\right]+$$
$$+\frac{1-\frac{\Lambda}{3}\left(\frac{3}{B^2}+\frac{3\bar\rho^2}{2}+\frac{B^2\bar\rho^4}{4}+\frac{B^4\bar\rho^6}{64}\right)}{\left(1+\frac{B^2\bar\rho^2}{4}\right)^2}\,\bar\rho^2\,d\bar\varphi^2$$

$$A_\mu = \left[0,0,0,\pm\frac{\bar\rho^2B/2}{1+B^2\bar\rho^2/4}\right]$$

Rotating cosmological electro-magnetic universe

$$ds^2 = \left(1+\frac{Q^2\rho^2}{4}\right)^2\left[-(dt-\omega d\varphi)^2 + dz^2 + \frac{\rho^2 d\rho^2}{\frac{k}{Q^4}\left(1+\frac{Q^2\rho^2}{4}\right) - 4Q^{-2} - \frac{4\Lambda}{3Q^4}\left(1+\frac{Q^2\rho^2}{4}\right)^4}\right]$$

$$+\frac{\frac{k}{Q^4}\left(1+\frac{Q^2\rho^2}{4}\right) - 4Q^{-2} - \frac{4\Lambda}{3Q^4}\left(1+\frac{Q^2\rho^2}{4}\right)^4}{\left(1+\frac{Q^2\rho^2}{4}\right)^2}(d\varphi - \omega dt)^2$$

$$A_\mu = \left[\frac{\omega\rho^2 B/2}{1+\frac{Q^2\rho^2}{4}} + E\ z,\ 0,\ 0,\ -\frac{\rho^2 B/2}{1+\frac{Q^2\rho^2}{4}} - \omega\ E\ z\right]$$

$$Q^2 = E^2 + B^2$$

when $k = 4(E^2+B^2) + 4\Lambda/3$ the conical singularity is avoided

BTZ black string in a electro-magnetic universe

$$ds^2 = \left(1 + \frac{\mathrm{B}^2\rho^2}{4}\right)^2 \left[(M - \lambda \mathrm{z}^2)dt^2 + \frac{d\mathrm{z}^2}{-M + \lambda \mathrm{z}^2} + \frac{\rho^2 d\rho^2}{\left(k_2 - \frac{\lambda\rho^2}{\mathrm{B}^2}\right)\left(1 + \frac{\mathrm{B}^2\rho^2}{4}\right) - 4\mathrm{B}^{-2} - \frac{4\Lambda}{3\mathrm{B}^4}\left(1 + \frac{\mathrm{B}^2\rho^2}{4}\right)^4} \right]$$

$$+ \frac{\left(k_2 - \frac{\lambda\rho^2}{\mathrm{B}^2}\right)\left(1 + \frac{\mathrm{B}^2\rho^2}{4}\right) - 4\mathrm{B}^{-2} - \frac{4\Lambda}{3B^4}\left(1 + \frac{\mathrm{B}^2\rho^2}{4}\right)^4}{\left(1 + \frac{\mathrm{B}^2\rho^2}{4}\right)^2} d\varphi^2$$

$$A_\mu = \left[0, 0, 0, \pm \frac{\rho^2 B/2}{1 + B^2\rho^2/4}\right]$$

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