Groups of Coordinate Transformations Between Accelerated Frames Based on the Equivalence Principle

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Georgy I. Burde

Jacob Blaustein Institutes for Desert Research, Ben-Gurion University
Sede-Boker Campus, 84990, Israel

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Motivation

The uniformly accelerated systems related by the equivalence principle to the homogeneous gravitational fields are the issues that are of fundamental importance for the understanding of general relativity. A discussion of accelerated systems is also essential to obtain a good understanding of special relativity. In both contexts, a description of events and particle motions from the point of view of an observer fixed in an accelerated reference frame is of the main interest. While the coordinate transformations from inertial to accelerated frames are widely studied in the literature from different physical approaches, there is very little discussion of transformations *between* accelerated frames although it is evident that, conceptually, such transformations should play a central role in the theory of accelerated frames. like as the Lorentz transformations do for inertial frames. An additional feature which makes studying the transformations between accelerated frames important is that, based on fundamental physical principles, they should possess the group property while for the transformations between inertial and accelerated frames, due to the distinguished role of the former, the group property is not expected.

Main principles

- **Group property**.
- **Invariance of the interval**:

$$ds^2 = u(z)^2 dt^2 - \left(\frac{u'(z)}{g}\right)^2 dz^2 - dx^2 - dy^2$$

$$u(z) \approx 1 + gz \quad \text{for} \quad gz \to 0$$

- The **equivalence principle** underlines the form of the interval. According to this principle, the metric in the accelerating coordinate system is the same as that of a uniform and constant gravitational field

Previous works

The Lie group techniques have been applied to the problem of determining transformations between accelerated frames in Bourgin (1936) and Hill (1945). Both works were stimulated by Page's "New Relativity"(Page, 1936) who defined transformations between "equivalent" accelerated frames on a purely kinematic basis and particularized the case of uniform acceleration.

Hsu and Hsu (1997), Hsu and Kleff (1998) derived transformations between accelerated frames using a new approach of *the limiting 4-dimensional symmetry*. They claim that transformations obtained using the conventional .gravitational approach based on the metric of static homogeneous gravitational field cannot be smoothly connected to the Lorentz transformations in the limit of zero accelerations since it involves only one parameter, i.e., acceleration, and, hence, do not reduce to the Lorentz transformation with a non-zero velocity in the limit of zero acceleration.

Group of transformations

One-parameter group of transformations between two accelerated frames $K(T, X, Y, Z, G)$ and $k(t, x, y, z, g)$:

$$g = h_0(G; \mathbf{a}), \quad t = h_1(T, Z, X, Y, G; \mathbf{a}), \quad z = h_2(T, Z, X, Y, G; \mathbf{a}),$$
$$x = h_3(T, Z, X, Y, G; \mathbf{a}), \quad y = h_4(T, Z, X, Y, G; \mathbf{a})$$

where $\mathbf{a}$ is a group parameter.

Change of variables $(t, z) \to (f_1, f_2)$:

$$f_1 = p\left(\sinh\mu + \cosh\mu\right), \quad f_2 = p\left(\sinh\mu - \cosh\mu\right)$$

where

$$p = \frac{u(z)}{g}, \quad \mu = gt$$

Condition of infinitesimal invariance

- *Infinitesimal transformations*

$$g \approx G + \mathbf{a}\,\chi(G),$$
$$f_1 \approx F_1 + \mathbf{a}\,\phi_1(F_1, F_2, X, Y, G), \quad f_2 \approx F_2 + \mathbf{a}\,\phi_2(F_1, F_2, X, Y, G),$$
$$x \approx X + \mathbf{a}\,\xi(F_1, F_2, X, Y, G), \quad y \approx Y + \mathbf{a}\,\eta(F_1, F_2, X, Y, G)$$

The infinitesimal operator (generator of the group)

$$\mathbf{X} = \phi_1 \frac{\partial}{\partial f_1} + \phi_2 \frac{\partial}{\partial f_2} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \chi \frac{\partial}{\partial g}$$

- *Condition of invariance*

$$ds^2 = dS^2 + \mathbf{a}\,\Delta, \quad ds^2 = dS^2 \;\Rightarrow\; \Delta = 0 \;\Rightarrow\; \text{determining equations}$$

- *Group generators – solutions of determining equations*

Finite transformations

Finite transformations are recovered by exponentiating the infinitesimal operator (generator of the group)

$$\mathbf{h} = e^{\mathbf{X}}\mathbf{H}$$

where $\mathbf{h}$ is a vector of variables $\{f_1, f_2, x, y, g\}$.

It is equivalent to solving the initial data problem (Lie equations with initial conditions):

$$\begin{aligned}
&\frac{df_1(a)}{da} = \phi_1(f_1(a), f_2(a), x(a), y(a), g(a)),\\
&\frac{df_2(a)}{da} = \phi_2(f_1(a), f_2(a), x(a), y(a), g(a)),,\\
&\dots\\
&\frac{dg(a)}{da} = \chi(g(a));\\
&f_1(0) = F_1,\ f_2(0) = F2,\ x(0) = X, \quad y(0) = Y,\ g(0) = G.
\end{aligned}$$

Solutions of determining equations for group generators

$$\phi_1(F_1, F_2, X, Y, G) = a_0 F_1 + 2\lambda_1 X - 2\lambda_0 Y + \frac{\lambda_4}{G},$$
$$\phi_2(F_1, F_2, X, Y, G) = -a_0 F_2 - 2\lambda_3 X - 2\lambda_2 Y + \frac{\lambda_5}{G},$$
$$\xi(F_1, F_2, X, Y, G) = \lambda_1 F_2 - \lambda_3 F_1 - a_9 Y + \frac{a_7}{G},$$
$$\eta(F_1, F_2, X, Y, G) = -\lambda_0 F_2 - \lambda_2 F_1 + a_9 X + \frac{a_8}{G},$$
$$\chi(G) = G$$

where

$$\lambda_0 = \frac{a_6 - a_5}{2},\ \lambda_1 = \frac{a_3 - a_4}{2},\ \lambda_2 = -\frac{a_5 + a_6}{2},$$
$$\lambda_3 = -\frac{a_3 + a_4}{2},\ \lambda_4 = a_1 - a_2,\ \lambda_5 = a_1 + a_2.$$

and all a_i, $i = 1, 2, \ldots, 9$, are nondimensional constants.

Multi-parameter groups

One-parameter groups are recovered by exponentiating the generator of the group

$$\mathbf{h} = e^{\mathbf{X}}\mathbf{H}$$

Since the generator includes arbitrary parameters it may be equally considered as defining a multi-parameter group of transformations. An r-parameter Lie group of finite transformations is defined by

$$\mathbf{h} = e^{a\sum_{\alpha=1}^{r} k_\alpha \mathbf{X}_\alpha}\mathbf{H},$$

or equivalently by

$$\mathbf{h} = \prod_{\alpha=1}^{r} e^{\sigma_\alpha \mathbf{X}_\alpha}\mathbf{H} = e^{\sigma_1 \mathbf{X}_1} e^{\sigma_2 \mathbf{X}_2} \cdots e^{\sigma_r \mathbf{X}_r}\mathbf{H}$$

Lie algebras

For the transformations to form an *r*-parameter Lie group of transformations the corresponding infinitesimal generators $\{\mathbf{X}_\alpha\}$, $\alpha = 1, 2, \ldots, r$, must form an *r*-dimensional *Lie algebra* which is a vector space with an additional law of combination of elements (the commutator)

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta] = \mathbf{X}_\alpha \mathbf{X}_\beta - \mathbf{X}_\beta \mathbf{X}_\alpha \quad (1)$$

with a property of closure with respect to commutation

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta] = C^\gamma_{\alpha\beta} \mathbf{X}_\gamma \quad (2)$$

where $C^\gamma_{\alpha\beta}$ are constants (structural constants). Having a set of infinitesimal operators, which satisfy that property, one can construct the corresponding *r*-parameter group of transformations as a composition of *r* one-parameter groups generated by each of base operators $\mathbf{X}_\alpha$ via exponentiation or, what is the same, solution of the Lie equations.

Two-dimensional (1+1) transformations

Specifications

- 1. The groups of transformations for which $x = X$ and $y = Y$ and transformations of z and t do not involve the variables x and y are considered.
- 2. The condition that the same event is a space-time origin of both frames is imposed.
- 3. The transformations including transformations to an inertial frame as a particular case (allowing nonsingular limit $g \to 0$) are separated.

Conditions 1 and 2 can be imposed on infinitesimals which yields

$$\phi_1(F_1, F_2, G) = a_0 F_1 - \frac{1 + a_0}{G}, \quad \phi_2(F_1, F_2, G) = -a_0 F_2 + \frac{1 - a_0}{G},$$
$$\chi(G) = G$$

Condition 3 can be used only after identifying finite transformations.

One-parameter groups

Solving the Lie equations yields

$$g = Ge^{a}, \quad f_1 = F_1 e^{a_0 a} + \frac{e^{-a} - e^{a_0 a}}{G}, \quad f_2 = F_2 e^{-a_0 a} - \frac{e^{-a} - e^{-a_0 a}}{G}.$$

where a is a group parameter and a_0 is a real number. It is easily checked that these transformations leave the interval invariant. To check whether the transformations allow a nonsingular limit $g \to 0$, a small parameter ϵ

$$\epsilon = \frac{g}{G}, \quad a = \ln \epsilon$$

is introduced and two first terms of expansions with respect to ϵ are calculated. It appears that for any real $a_0 \neq 0$ the first terms of the expansions for z and t contain singularities so that the one-parameter group of transformations may include a transformation to an inertial frame as a particular case (or, in other terms, the value of the group parameter $a = -\infty$ is eligible) only if $a_0 = 0$.

One-parameter group of two-dimensional (1+1) transformations

$$\mu = \operatorname{arccoth} \frac{\frac{1}{g} - \frac{1}{G} + P \cosh M}{P \sinh M}$$

$$p = \sqrt{\left(\frac{1}{g} - \frac{1}{G}\right)^2 + P^2 + 2\left(\frac{1}{g} - \frac{1}{G}\right) P \cosh M}$$

$$g = Ge^a$$

where

$$p = \frac{u(z)}{g},\ \mu = gt; \quad P = \frac{u(Z)}{G},\ M = GT$$

In the limit of $g \to 0$, expanding the right-hand sides of the expressions up to the order of $\epsilon = g/G$ yields the transformations to an inertial frame

$$t' = P \sinh M, \quad z' = -\frac{1}{G} + P \cosh M$$

Two-parameter group

In terms of f_1, f_2, g

$$f_1 = e^b F_1 + \frac{e^{-a} - e^b}{G}, \quad f_2 = e^{-b} F_2 + \frac{e^{-b} - e^{-a}}{G}, \quad g = Ge^a$$

where (a, b) are the group parameters. The transformations satisfy the group property with the addition law of composition for both group parameters a and b.

In terms of z, t, g

$$\mu = \mathrm{arccoth} \frac{G - g \cosh b + gGP \cosh(M + b)}{g(-\sinh b + GP \sinh(M + b))},$$

$$p = \sqrt{\frac{1}{g^2} + \frac{1}{G^2} + P^2 - \frac{2(gP \cosh M + \cosh b - GP \cosh(M + b))}{gG}},$$

$$g = Ge^a$$

$$p = \frac{u(z)}{g},\ \mu = gt; \quad P = \frac{u(Z)}{G},\ M = GT$$

Physical meaning of the parameter b

The physical meaning of the parameter b becomes clear when one calculates the relative velocity of the space origins of the frames k and K at the initial moment $t = T = 0$ when the origins of both frames coincide. However, to do it the function $u(z)$ is to be specified. The Møller metric and Lass metric were considered.

$$u(z) = 1 + gz \; (\text{Møller metric}) \quad u(z) = e^{gz} \; \text{Lass metric}$$

For both metrics, calculations give for the velocity V of the origin of k measured by an observer at the origin of K the following

$$V = -\tanh b$$

Transformations in terms of V

$$\mu = \operatorname{arccoth}\frac{G+\gamma g\left(-1+GP\left(\cosh M - V\sinh M\right)\right)}{\gamma g\left(V+GP\left(-V\cosh M+\sinh M\right)\right)}, \quad \gamma = \frac{1}{\sqrt{1-V^2}},$$

$$p = \sqrt{\frac{1}{g^2}+\frac{1}{G^2}+P^2-\frac{2\gamma}{gG}+\left(-\frac{2P}{G}+\frac{2P\gamma}{g}\right)\cosh M - \frac{2PV\gamma\sinh M}{g}}$$

where

$$p=\frac{u(z)}{g},\ \mu = gt; \quad P=\frac{u(Z)}{G},\ M=GT$$

Taking the limit of both small g and G yields **the Lorentz transformations**

$$z=\frac{Z-VT}{\sqrt{1-V^2}}, \quad t=\frac{T-VZ}{\sqrt{1-V^2}}$$

Transformations to an inertial frame are obtained as a limit of $g\to 0$ as

$$z'=\frac{\gamma}{G}\left(-1+GP\left(\cosh M - V\sinh M\right)\right)$$

$$t'=\frac{\gamma}{G}\left(V+GP\left(\sinh M - V\cosh M\right)\right)$$

Three-dimensional (2+1) and four-dimensional(3+1) transformations

In the case of **three-dimensional transformations**, after satisfying the requirement that transformations to an inertial frame were included as a particular case, only a two-parameter group remains. The transformations belonging to that group correspond to the situation when the direction of the relative velocity of the frame space origins at the initial moment is not along the z-axis. In the limit when both frames are inertial, the transformations become the Lorentz boost in arbitrary direction.

In the case of **four-dimensional transformations**, also there remains only a two-parameter group, and the corresponding transformations can be reduced to the three-dimensional case by the coordinate change. Thus, the two-parameter group of three-dimensional transformations may be considered as a generalization of the Lorentz transformations to accelerated frames in four dimensions.

Applications: differential aging between accelerated twins

Both twins are accelerated but Alice moves with uniform constant acceleration g and Bob (returning twin) changes his acceleration G several times. Four steps:

- 1. $G = G_1 > g$; Bob's velocity changes from 0 to V.
- 2. $G = -G_1$; Bob's velocity changes from V to 0.
- 3. $G = -G_1$; Bob's velocity changes from 0 to $-V$.
- 4. $G = G_2$; Bob's velocity changes from V to the Alice velocity and Bob meets Alice

Conclusions

- The Lie group analysis is applied to determine groups of transformations between accelerated systems. The analysis is based on the equivalence principle according to which the metric in an accelerated frame has the form of static homogeneous gravitational field. Groups of transformations between accelerated frames are found from the condition of invariance of the interval under the transformations.
- For two-dimensional (1+1) transformations, the general result of the analysis is the two-parameter group of transformations which relate accelerated frames with nonzero relative velocity of the space origins at the initial moment. The transformations satisfy the desired property of reducing to the Lorentz transformations when accelerations of both frames vanish. and so may be considered as a proper generalization of the Lorentz boost to accelerated frames.

Conclusions

- In this respect, an argument, that can be found in the literature, that the transformations obtained using the "gravitational approach" based on the metric of static homogeneous gravitational field cannot be smoothly connected to the Lorentz transformations in the limit of zero accelerations due to the lack of a velocity parameter in the metric, seems unfounded. The analysis of the present paper shows that the velocity parameter does not need to appear in the metric in order to take part in the transformations. It arises as an additional group parameter in the transformations derived through the multi-parameter group analysis.
- The most general transformations between accelerated frames having nonzero relative velocity at the initial moment are given by the three-dimensional (2+1) transformations which represent a generalization of the Lorentz transformations to accelerated frames for the case when the initial relative velocity of the frames is not along the direction of acceleration.