Canonical Quantization of Constrained Lagrangians and Conditional (Nöther) Symmetries

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Outline

Prototype Lagrangian of minisuperspace models of Einstein Gravity

$$L = \frac{1}{2N} G_{\alpha\beta}(q)\, \dot{q}^{\alpha}\, \dot{q}^{\beta} - N\, V(q) \qquad (1)$$

$$q^{\alpha}(x) \quad , \quad \alpha = 1, ..., n$$

x may either play the role of time, in Cosmological Geometries, or of the radial coordinate in Point-like metrics

$$P_{\alpha} = \frac{\partial L}{\partial \dot{q}^{\alpha}} = \frac{1}{N} G_{\alpha\beta}\, \dot{q}^{\beta} \qquad P_N \approx 0 \qquad \text{(Primary Constraint)}$$

$$H = P_\gamma \, \dot{q}^\gamma - L = N \left(\frac{1}{2} \, G^{\alpha\beta}(q) \, P_\alpha \, P_\beta + V(q) \right) = N \, X \qquad (2)$$

$$\dot{P}_N := \{P_N, H\} \approx 0 \Rightarrow \qquad X := \frac{1}{2} \, G^{\alpha\beta}(q) \, P_\alpha \, P_\beta + V(q) \approx 0$$

(X Secondary Constraint)

$\{P_N, X\} \approx 0 \Rightarrow P_N, X$ First Class Constraints

Conditional Symmetries

$$\mathcal{L}_\xi X = \phi(q) X \tag{3}$$

$$\mathcal{L}_\xi G^{\alpha\beta} = \phi G^{\alpha\beta} \qquad \mathcal{L}_\xi V = \phi V \tag{4}$$

ξ remain form invariant under a scaling

$$G^{\alpha\beta} \to \omega(q) G^{\alpha\beta} \quad , \quad V \to \omega(q) V$$

$$\mathcal{L}_\xi \left(\omega G^{\alpha\beta}\right) = \omega \mathcal{L}_\xi G^{\alpha\beta} + G^{\alpha\beta} \mathcal{L}_\xi \omega = \left(\phi + \xi^\alpha \frac{\omega_{,\alpha}}{\omega}\right) \omega G^{\alpha\beta}$$

$$\mathcal{L}_\xi \left(\omega V\right) = \omega\, \mathcal{L}_\xi V + V\, \mathcal{L}_\xi \omega = \left(\phi + \xi^\alpha \frac{\omega_{,\alpha}}{\omega}\right) \omega V$$

$Q_I := \xi_I^\alpha P_\alpha \Rightarrow$ conserved quantities $\{\xi_I^\alpha P_\alpha, H\} \approx 0$ $\quad I = 1, ..., m \quad \left(\leq \frac{n(n+1)}{2}\right)$

on mass shell $\qquad Q_I := \xi_I^\alpha P_\alpha = \kappa_I = constant$

Quantization : Schrödinger picture

$$\hat{P}_N \Psi := -i \frac{\partial}{\partial N} \Psi = 0 \Rightarrow \Psi(q)$$

$$\hat{X} \Psi = 0 \Rightarrow \left[-\frac{1}{2\mu} \partial_\alpha (\mu \, G^{\alpha\beta} \partial_\beta) + V(q) \right] \Psi = 0$$

natural measure $\mu = \sqrt{|\det G_{\alpha\beta}|}$

$$(\hat{Q}_I - \kappa_I) \Psi = 0$$

$$\hat{Q}_I \Psi \equiv -\frac{i}{2\mu} \left(\mu \, \xi_I^\alpha \partial_\alpha + \partial_\alpha \mu \xi_I^\alpha \right) \Psi$$

The consistency requirements $[\hat{Q}_I - \kappa_I, \hat{Q}_J - \kappa_J]\Psi = 0$ imply the selection rule

$$i\, C^M_{IJ} \kappa_M - \frac{1}{2}\left(\mathcal{L}_{\xi_I} \nabla \cdot \xi_J - \mathcal{L}_{\xi_J} \nabla \cdot \xi_I + C^M_{IJ}\, \nabla \cdot \xi_M\right) = 0$$

maximal Abelian subgroup + Casimir invariants

Specific Example : Schwarzschild metric

$$ds^2 = -a(r)^2 dt^2 + N(r)^2 dr^2 + b(r)^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \quad (5)$$

$$L = \frac{2a}{N}\dot{b}^2 + \frac{4b}{N}\dot{a}\dot{b} + 2aN \quad (6)$$

valid Lagrangian: E-L equations $\Leftrightarrow G_{\mu\nu} = 0$

$$H = N\left(-\frac{a}{8b^2}P_a^2 + \frac{1}{4b}P_a P_b - 2a \right) \quad (7)$$

$$X = -\frac{a}{8b^2}P_a^2 + \frac{1}{4b}P_a P_b - 2a \approx 0 \quad (8)$$

$$Q_I = \xi_I^\alpha P_\alpha \quad \alpha = a, b$$
$$Q_1 = -aP_a + bP_b$$
$$Q_2 = \frac{1}{ab} P_a$$
$$Q_3 = -\frac{a}{2b} P_a + P_b$$

$$\{Q_1, X\} = X\,, \quad \{Q_2, X\} = -\frac{1}{a^2 b} X\,, \quad \{Q_3, X\} = \frac{1}{2b} X$$

$$\{Q_1, Q_2\} = -Q_2 \tag{9a}$$
$$\{Q_1, Q_3\} = Q_3 \tag{9b}$$
$$\{Q_2, Q_3\} = 0 \tag{9c}$$

On the Schwarzschild metric: $Q_1 = 4M$, $Q_2 = 2c$, $Q_3 = \frac{4}{c}$,

$$\begin{aligned} M &\longrightarrow \text{essential} \\ c &\longrightarrow \text{absorbable} \end{aligned}$$

From (9) $\Rightarrow \{Q_1, Q_2 Q_3\} = 0 \quad \Rightarrow$ $Q_2 Q_3$ Casimir Invariant

$$Q_2 Q_3 = 8 \qquad Q_2 Q_3 - 8 = \tfrac{4}{a} X \equiv \tilde{X}$$

under a scaling

$$N \rightarrow \frac{\tilde{N}}{a} : \qquad Q_2 Q_3 - 8 \equiv 4\tilde{X}, \qquad \tilde{G} = \sqrt{|\det \tilde{G}_{\alpha\beta}|} = 4ab$$

$$\hat{\tilde{X}} := -\frac{1}{2\sqrt{\tilde{G}}} \partial_\alpha \left(\sqrt{\tilde{G}}\, \tilde{G}^{\alpha\beta}\, \partial_\beta \right) - 2$$

$$\hat{Q}_1 := -i \left(\xi_1^\alpha \partial_\alpha + \frac{1}{2\sqrt{\tilde{G}}} \partial_\alpha \left(\sqrt{\tilde{G}} \xi_1^\alpha \right) \right)$$

Imposition of the equations

$$\hat{\tilde{X}}\Psi = 0$$

$$\hat{Q}_1 \Psi = 4M\Psi$$

leads to

$$\Psi = a^{-i4M}\omega(u) \quad \text{with} \quad u = a\,b$$

and ω solution of $\quad u^2\,\omega''(u) + u\,\omega'(u) + 16\,(u^2 + M)\,\omega(u) = 0$

$$\omega(u) = J_{i4M}(4u)$$

probability ?

$$\int \mu(a,b)\psi^*\psi \, dadb = \int uJ_{i4M}(4u)^* J_{i4M}(4u)\, du$$

Invariant characterization of different geometries contained in (5)

$$u = F(b)$$

Probability of the configuration b within the specific geometry F

$$P_F = \frac{F(b)J_{i4M}(4F(b))^* J_{i4M}(4F(b))}{\int uJ_{i4M}(4u)^* J_{i4M}(4u)\, du}$$