Construction of dynamical vacuum black hole spacetimes from scattering data

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Outline

1. Background: The non-linear stability of Kerr problem
2. Construction of dynamical black holes from scattering data: statement of the theorem
3. Why the stability problem is hard
4. Why the scattering problem is easier–*but not trivial*

1. The nonlinear stability of Kerr problem

The Schwarzschild and Kerr families

Recall the 2-parameter **Kerr family** of **stationary, axisymmetric** vacuum (i.e. $\mathrm{Ric} = 0$) spacetimes $(\mathcal{M}, g_{M,a})$ first discovered in 1963. The parameters are called **mass** M and **specific angular momentum** a, i.e. angular momentum per unit mass. The case $a = 0$ is known as Schwarzschild (1916). In local coordinates:

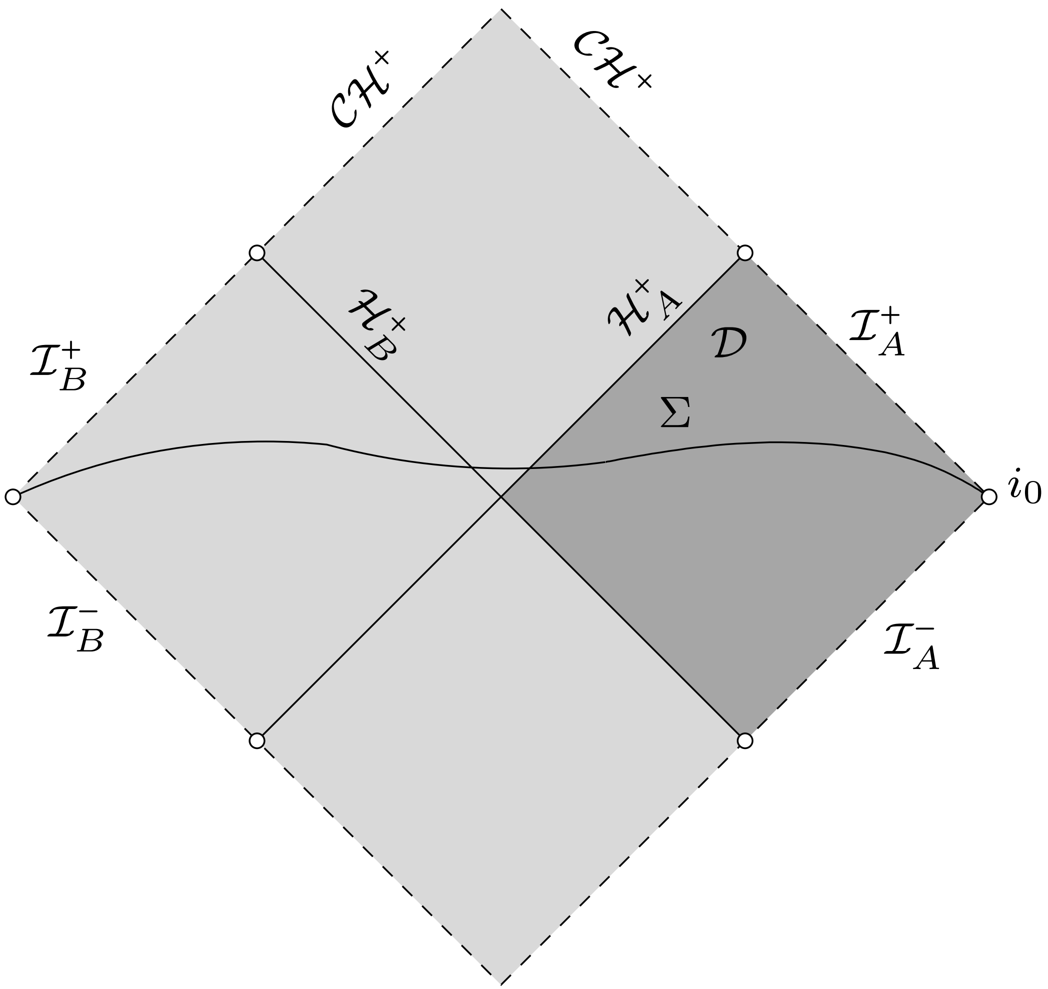
$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a\,dt - (r^2 + a^2)d\phi\right)^2$$

$\rho^2 = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2 = (r - r_-)(r - r_+),\ |a| \le M \implies 0 \le r_- \le r_+$

$|a| < M$ **subextremal** black hole, $|a| = M$ **extremal** black hole,

$|a| > M$ "naked singularity" case

"Penrose diagram" for Kerr $0 < |a| < M$



Uniqueness of the Kerr family as stationary, black hole solutions

The Kerr black holes are thought to be the unique family of **stationary**, black hole solutions of the Einstein vacuum equations.

Under the assumption of **real analyticity**, this has been shown putting together work of HAWKING, CARTER, ROBINSON, PAPAPETROU (see also recent improvements of CHRUŚCIEL–COSTA).

The assumption of analyticity is incompatible with causality. In recent work of IONESCU–KLAINERMAN, ALEXAKIS–IONESCU–KLAINERMAN, it has been shown that there are no other **smooth** subextremal stationary vacuum black holes solutions *near the Kerr family*.

This latter result is sufficient for our purposes here, as it means that it makes sense to entertain the notion of *asymptotic stability of the Kerr family*.

The nonlinear stability problem of Kerr

Perturbations of a (subextremal) Kerr metric should dynamically approach the Kerr family **in the exterior-to-the-black-hole region**:

Conjecture (Stability of Kerr)**.** *Let* $(\Sigma, \bar{g}, K)$ *be a vacuum initial data set sufficiently close to the initial data on a Cauchy hypersurface in the Kerr solution* $(\mathcal{M}, g_{M_i,a_i})$ *for some* **subextremal** *parameters* $0 \le |a_i| < M_i$*. Then the maximal Cauchy development* $(\mathcal{M}, g)$ *of the data under evolution by the vacuum equations*

$$\mathrm{Ric} = 0$$

possesses a complete null infinity $\mathcal{I}^+$ *such that the metric restricted to* $J^-(\mathcal{I}^+)$ *approaches a Kerr solution* $(\mathcal{M}, g_{M_f,a_f})$ *in a uniform way with quantitative decay rates, where* $|a_f| < M_f$ *are near* a_i*,* M_i *respectively.*

Note: $a_i = 0$ **will not imply** that $a_f = 0$!

Cf. non-linear stability of Minkowski space (Christodoulou–Klainerman).

2. Construction of dynamical black holes from scattering data: statement of the theorem

Theorem (M.D., G. Holzegel, I. Rodnianski). *Given suitable smooth scattering "data" on the horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$, asymptoting to the induced Kerr geometry with parameters $|a| \leq M$, then there exists a corresponding smooth vacuum black hole spacetime asymptotically approaching in its exterior region the Kerr solution with parameters a and M.*

Corollary. *There exist black hole spacetimes with smooth horizon $\mathcal{H}^+$ and complete null infinity $\mathcal{I}^+$ which are not exactly Schwarzschild or Kerr.*

Remarks on the statement

1. The set of solutions is parametrized by “a full set” of scattering data. Thus, the class of solutions is “large” in this sense.

2. The assumptions of the theorem will require however that the scattering data decay exponentially *along* $\mathcal{H}^+$, $\mathcal{I}^+$. This is in contrast to the expected behaviour of the “generic” solution of the forward problem, where decay along $\mathcal{H}^+$ and $\mathcal{I}^+$ is expected to be inverse polynomial.

3. Nonetheless, for reasons we shall see, the restriction to exponentially decaying data along $\mathcal{H}^+$ and $\mathcal{I}^+$ is expected to be necessary for the type of formulation as in the Theorem.

3. Why the stability problem is hard

The difficulties of the stability problem can be seen to enter at three levels:

3.1. The "poor man's" linearisation: $\Box_g \psi = 0$.

3.2. The equations of linearised gravity.

3.3 The nonlinear Einstein equations.

3.1 The poor man's linearisation: $\Box_g \psi = 0$ on Kerr

Classical work: Regge–Wheeler 1957, Carter 1968, Price 1972, Wald 1979, Whiting 1982, Kay–Wald 1986 (*heuristic mode analysis, boundedness for Schwarzschild*)

M.D.–Rodnianski, Tataru–Tohaneanu, Andersson–Blue, Aretakis

(*quantitative bounds for the global behaviour of solutions to the wave equation on Kerr* $|a| \le M$)

See also related work with cosmological constant:

$\Lambda > 0$: M.D.–Rodnianski, Bony-Häfner, Melrose–Sa Barreto–Vasy, Vasy, Zworski–Sa Barreto, Dyatlov, Schlue

$\Lambda < 0$: Holzegel, Holzegel–Smulevici

Phenomena

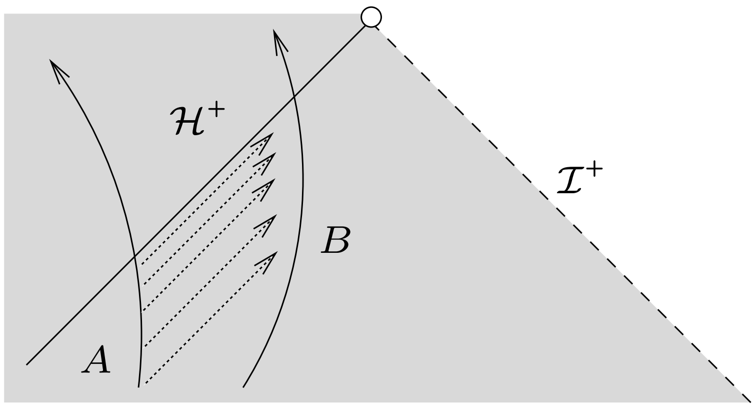
1. Red-shift
2. Superradiance
3. Trapping
4. Phenomena 1.–3. are strongly coupled as $|a| \to M$.

In fact, the stability result breaks down exactly at $|a| = M$. (Aretakis).

5 Behaviour near $\mathcal{I}^+$

The red-shift

The redshift is classically understood in the geometric optics approximation in terms of signals sent and received by two observers A and B, respectively.



First discussed in the Schwarzschild setting by OPPENHEIMER–SNYDER, 1939.

In fact, properly thought of, only depends on positivity of surface gravity.

The red-shift is a stability mechanism!

Extremal case $a = M$: <u>The red-shift factor at the horizon vanishes.</u>

Superradiance

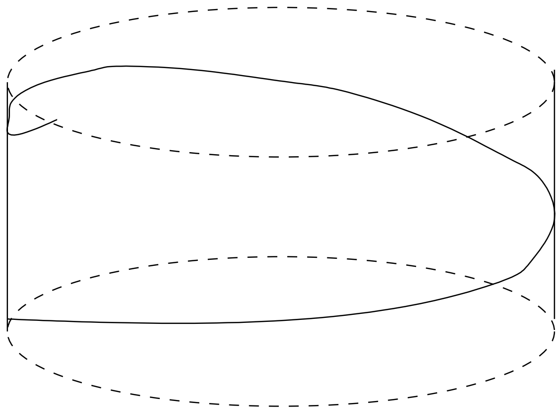
In Schwarzschild ($a = 0$), the Killing vector field ∂_t is timelike in the exterior, becoming null on the horizon. Thus there is a **conserved** *(by Noether)* **non-negative definite** *(by the timelike condition)* energy. The only subtlety is that this energy degenerates at the horizon.

In stationary perturbations of Schwarzschild, ∂_t in general becomes **spacelike** near the horizon. This happens in particular for Kerr for **all** $0 \neq |a| \leq M$. The corresponding energy is conserved but does not have a sign. For particle motion, this leads to the so-called Penrose process. For waves, this leads to the phenomenon of *superradiance* (Zeldovich).

In particular, using the conservation law associated to ∂_t one cannot prove *a priori* boundedness, even away from the horizon.

Trapping

On Schwarzschild, the "photon sphere" $r = 3M$ has the property that it contains null geodesics. These null geodesics thus neither escape to $\mathcal{I}^+$ nor to the horizon $\mathcal{H}^+$.



In Kerr, the behaviour persists, but it is more complicated!

One can concentrate energy for arbitrarily large times near trapped null geodesics. One has to capture this to prove dispersive results.

In particular, pointwise-in-time decay estimates for energy must lose derivatives (Ralston).

3.2 The equations of linearised gravity.

When one linearises the Einstein equations around the trivial solution Minkowski space, say in harmonic coordinates, then each linearised metric component indeed satisfies

$$\Box_g h^{\mu\nu} = 0. \tag{1}$$

When one linearises however, around a nontrivial solution like Kerr, the linearised system has highly non-trivial tensorial structure. This gives rise to additional difficulties not present in (1).

1. No obvious Lagrangian structure, thus no a priori conserved non-negative energy, even in Schwarzschild where ∂_t is causal.
2. Not all degrees of freedom decay, for in particular, linearisation must see nearby Kerr's. What is the mechanism that keeps these bounded?

3.3 The nonlinear Einstein equations

The difficulties entering at the level of the nonlinearity include of course the familiar difficulties which are already manifest in stability of Minkowski space (Christodoulou–Klainerman).

1. Quadratic nonlinearities in derivatives of the metric, plus quasilinearity. Need special structure to ensure even local existence at $\mathcal{I}^+$.

2. To uncover this structure, need to introduce an elaborate gauge, where wave equations for curvature are coupled with transport and elliptic equations for the connection.

In view, moreover, of the additional difficulties described previously, we could add:

3. How do these difficulties interact with the difficulties of 3.1–3.2?

4. How does one pick the final parameters a, M, and the centre of mass frame?

cf. Holzegel

4. Why the scattering problem is easier–but not trivial!

4.1 The scalar wave equation $\Box_g \psi = 0$

Dimock scattering theory for $\Box_g \psi = 0$ on Schwarzschild

Recall the Killing field ∂_t in Schwarzschild.

Let $\mathcal{X}_0$ denote the space of finite energy flux with respect to ∂_t on a slice $t = 0$ of the exterior.

Let $\mathcal{X}_{\mathcal{I}^+}$ denote the space of finite asymptotic energy flux with respect to ∂_t on $\mathcal{I}^+$.

Let $\mathcal{X}_{\mathcal{H}^+}$ denote the space of finite energy flux with respect to ∂_t on $\mathcal{H}^+$. Note that this is highly degenerate!

Then we have:

Theorem (Dimock 1985). *The map $\mathcal{X}_0 \to \mathcal{X}_{\mathcal{I}^+} \oplus \mathcal{X}_{\mathcal{H}^+}$ defined by solving the forward problem and "restricting" to $\mathcal{H}^+$ and $\mathcal{I}^+$ is in fact an isomorphism.*

See also Bachelot.

This scattering theory, however, unfortunately does not go very far!

On Kerr, the forward map

$$\mathcal{X}_0 \to \mathcal{X}_{\mathcal{I}^+} \oplus \mathcal{X}_{\mathcal{H}^+}$$

is not even well defined.

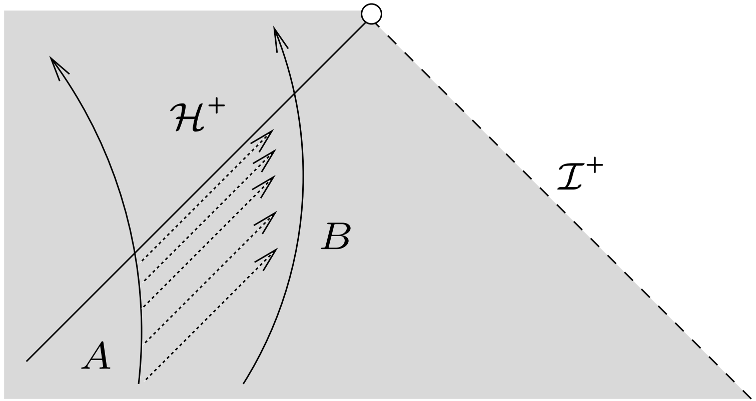
Uniform boundedness is only known for solutions with finite *non-degenerate* positive definite energy.

This is the energy associated with the vector field N related to the red-shift estimate.

Thus the ∂_t-scattering theory is inappropriate even for the scalar wave equation!

N-scattering theory for $\Box\psi = 0$ on Schwarzschild/Kerr

The **red-shift** is now a **blue-shift**.



This means that one *must* impose exponential decay along $\mathcal{H}^+$ and $\mathcal{I}^+$.

Once one accepts this obstruction, and imposes such data, then the scattering problem for $\Box_g \psi = 0$ becomes very easy!

One just needs to show that solutions grow at most exponentially when solving backwards. For this, one need only apply the energy identity for N, and Gronwall.

In particular, the trapping difficulty, so painful for the forward problem, does not appear.

4.2 Linearised gravity

Recall the characteristic new difficulties in passing from the scalar problem to linearised gravity.

In some sense, the first difficulty (lack of a conserved energy) is not relevant since we are imposing exponential decay and dealing with the N-energy.

On the other hand, the second difficulty remains. Not all degrees of freedom decay, and we need to prevent the non-decaying degrees of freedom from being infinitely **blue-shifted**.

Thus, one must still understand how to "separate out" the degrees of freedom which decay from those that don't, without destroying the structure of the equations.

4.3 The nonlinear vacuum Einstein equations

Again, recall the difficulties described in the forward problem.

We still have difficulties 1.–3.

(Note that, the blue-shift aside, difficulty 1. again would exclude a scattering theory based solely on the finiteness of the ∂_t flux on null infinity $\mathcal{I}^+$. Even to solve locally around null infinity, one must take weighted estimates, and this will require also decay *along* null infinity, though this obstruction will only be polynomial.)

One slide summary of the proof

We introduce a systematic formulation of a set of

- "renormalised" spin coefficients "Γ", and
- curvature coefficients "ψ"

such that $\Gamma = \psi = 0$ for Kerr. These are defined with respect to a null frame adapted to a double null foliation (Christodoulou).

The ψ satisfy Bianchi-type equations (hyperbolic) and Γ transport and elliptic equations. The structure of the system is preserved by commutation with respect to an appropriate set of commutation vector fields.

We apply energy estimates to ψ associated with N and with a new hierarchy of weighted vector fields near $\mathcal{I}^+$ capturing peeling. We apply transport estimates to control Γ. The weighted hierarchy also captures the "null condition".

$\Longrightarrow$ these weighted energies grow at most exponentially when solving backwards.

Some technical details

Ambient differential structure.

Approximation by a finite problem.

Prescription of data on null hypersurfaces. Constraints. (See Christodoulou).

Limit to null infinity. (See Christodoulou)

Well posedness and inherent loss of derivatives of characteristic initial value problem. (Rendall, Muller Zum Hagen, Christodoulou)

Differences of solutions.

Conjecture. *Consider scattering data which decays inverse polynomially along $\mathcal{I}^+$ and $\mathcal{H}^+$. Then one can attach a development spacetime $(\mathcal{M}, g)$, but, for generic such scattering data, $\mathcal{H}^+$ will be singular in the transverse directions.*

cf. Robinson–Trautman spacetimes

This conjecture should not be interpreted as suggesting that generic solutions of the forward problem cannot have polynomial decay! Rather, that one cannot "spot" the solutions of the forward problem arising from smooth data just by looking at the decay on $\mathcal{I}^+$ and $\mathcal{H}^+$.

Kerr-de Sitter? Here, a pure harmonic coordinate approach could work.

cf. also parametrizing solutions from scattering data for asymptotically pure de Sitter (H. Friedrich)