The CFC approximation in uniformly and differentially rotating relativistic stars

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Introduction

Evaluation of the conformal flatness condition (CFC) approximation in various physical systems is crucial in order to better understand the limits within which it can be applied, as well as the magnitude of the error one is making by choosing to apply the CFC approximation in a physical problem instead of full general relativity. Cook, Shapiro and Teukolsky (CST) [1] used the scheme developed by Komatsu, Eriguchi and Hachisu (KEH) [2, 3] in order to test the CFC approximation for the case of single, uniformly rotating, relativistic stars yielding very encouraging results. In this work, we study the quality of CFC for the case of differential rotation and we verify its excellent performance for the case of uniform rotation.

Following KEH the line element for a stationary, rotating, axisymmetric star in equilibrium is given by

$$ds^2 = -e^{\gamma+\rho}dt^2 + e^{\gamma-\rho}r^2\sin^2\theta(d\phi-\omega dt)^2 + e^{2\mu}(dr^2+r^2d\theta^2)\ , \quad (1)$$

where γ, ρ, ω and μ are metric potentials depending only on r and θ. We assume that the stellar matter behaves as a perfect fluid and that the equation of state obeys the polytropic relation

$$p = K\rho^{1+\frac{1}{N}}\ , \quad (2)$$

where ρ is the rest mass density, K the polytropic constant and N the polytropic index. For the case of differential rotation we adopt the same rotation law as in KEH, namely

$$F(\Omega) = A^2(\Omega_c - \Omega)\ , \quad (3)$$

where, A is a positive constant that determines the length scale over which the angular velocity changes within the star and Ω_c is the angular velocity at the center of the configuration.

Method

Using the basic assumption of the CFC approximation, $\gamma_{ab} = \psi^4 n_{ab}$, and the fact that for an axisymmetric star in spherical coordinates β^ϕ is the only non-zero component of the shift vector β^α, the line element in the CFC approximation is written as

$$ds^2 = -\alpha^2dt^2 + \gamma_{ij}(dx^i+\beta^i dt)(dx^j+\beta^j dt) \Rightarrow$$
$$ds^2 = -\alpha^2dt^2 + \psi^4(dr^2+r^2d\theta^2) + \psi^4r^2\sin^2\theta(d\phi+\beta^\phi dt)^2 \quad (4)$$

Comparing the above expression to the form of the line element in the KEH scheme, i.e relation (1), we get

$$\alpha = e^{(\gamma+\rho)/2}\ , \qquad \psi = e^{\mu/2} = e^{(\gamma-\rho)/4}\ , \qquad \beta^\phi = -\omega\ .$$

From the second relation above we obtain

$$\mu = \frac{\gamma-\rho}{2}\ . \quad (5)$$

Using the above relation in the standard KEH scheme instead of the differential equation for μ, we impose the CFC on our solution.

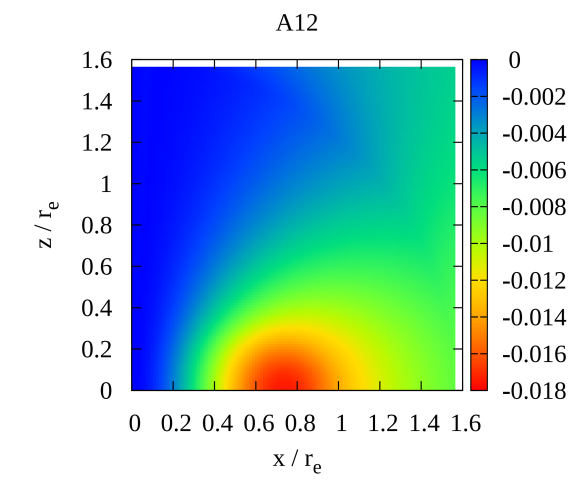
Various physical quantities are calculated for each model both in CFC and in full GR. As a diagnostic of the quality of the CFC we use the quantity

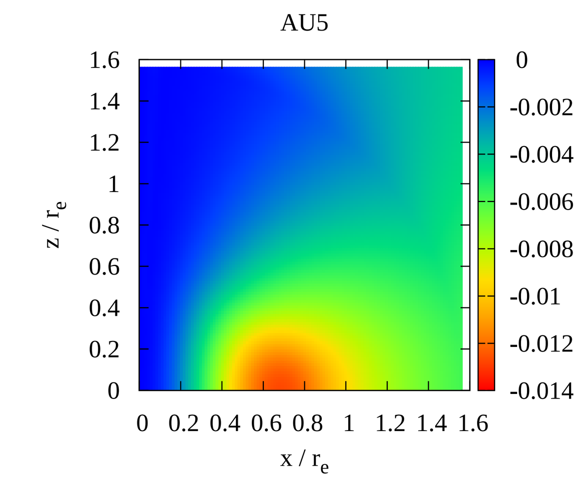
$$\Delta c = \frac{\mu_{\text{full GR}} - \mu_{\text{CFC}}}{\mu_{\text{CFC}}} = \frac{\mu - \frac{\gamma-\rho}{2}}{\mu}\ . \quad (6)$$

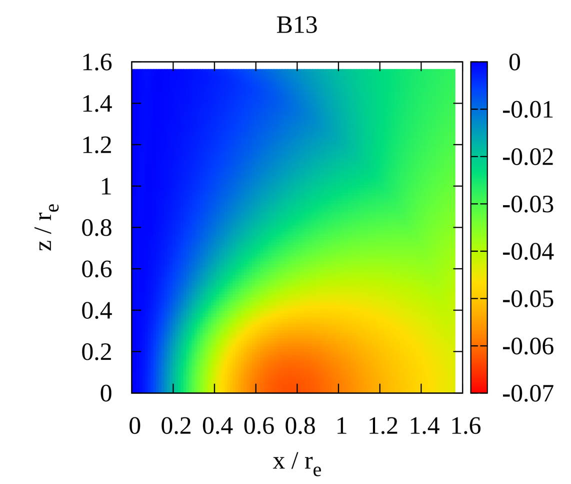
In addition, relative differences between CFC and full GR are calculated for every physical quantity. The above procedure is applied in the sequences of models that appear in Table I of [4]. Sequences A and B consist of differentially rotating models, whereas sequences AU and BU of uniformly rotating models. Configurations in sequences A and AU have constant rest mass of $M_0 = 1.506M_\odot$ and configurations in sequences B and BU have constant central mass density of $\rho_c = 1.28\times10^{-3}$ or equivalently constant central energy density of $\epsilon_c = 1.444\times10^{-3}$.

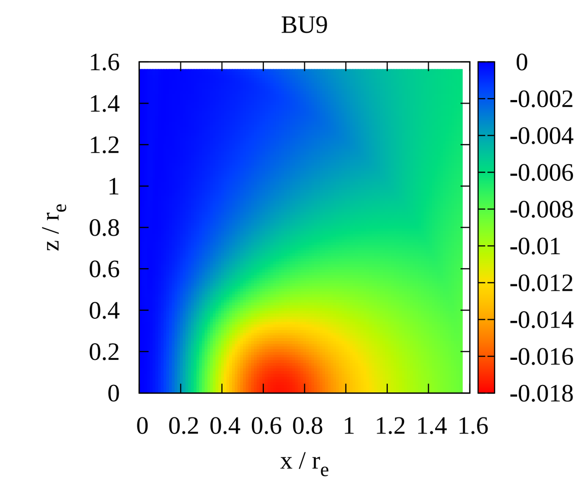
Results

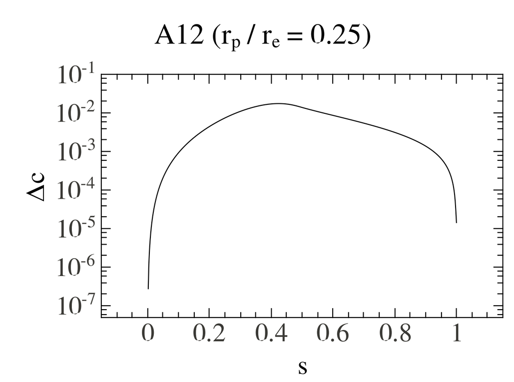
The outcome of our tests for the different models constructed is very encouraging. Concerning the physical quantities that we calculated, the relative differences between CFC and full GR were around or well below 10^{-2} in most cases. We also present the quantity Δc in the $x-z$ plane and also as a function of the CST variable s which maps radial infinity to $s = 1$, for the fastest rotating models of each sequence. The polar to equatorial radius ratio, r_p/r_e, is used to indicate how fast a certain configuration is rotating. The most extreme case is the model $B13$ but even in that case the maximum value of Δc is around 6%. The following figures summarize our results.

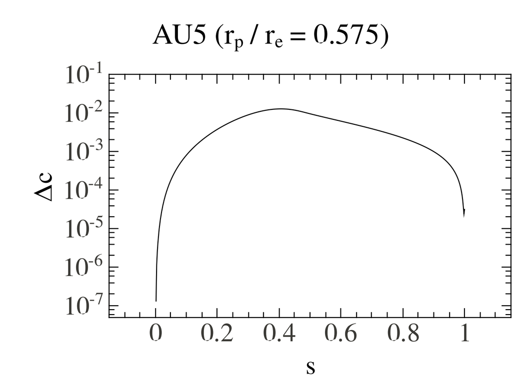


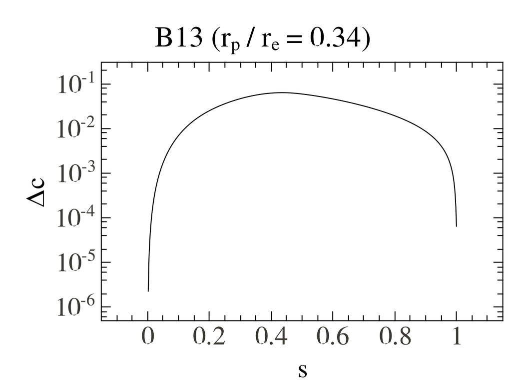


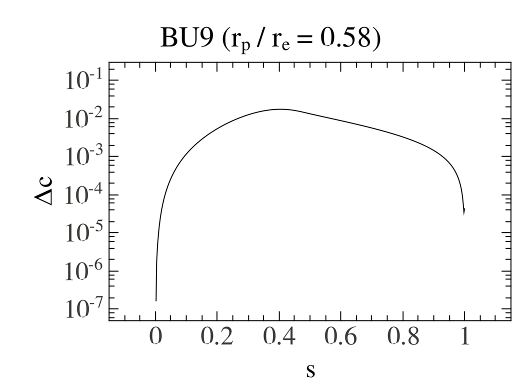


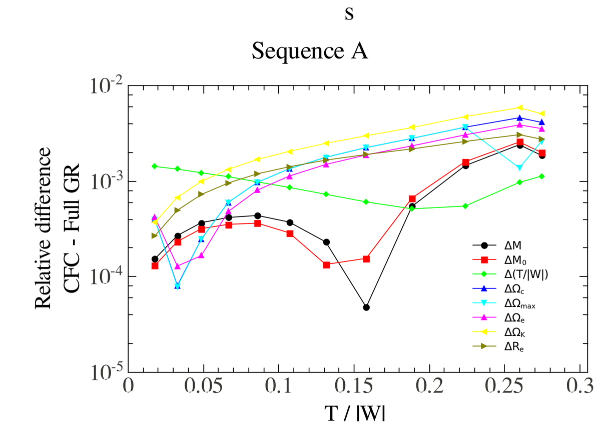


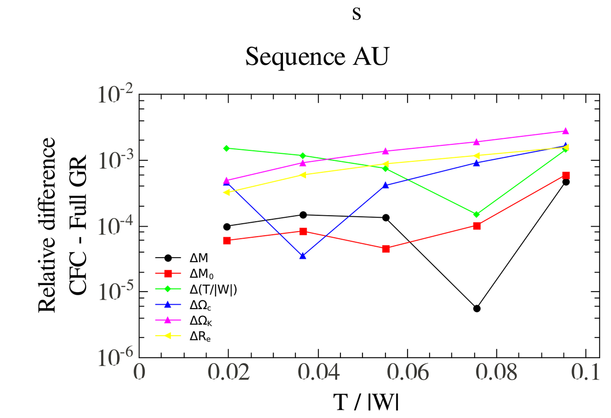


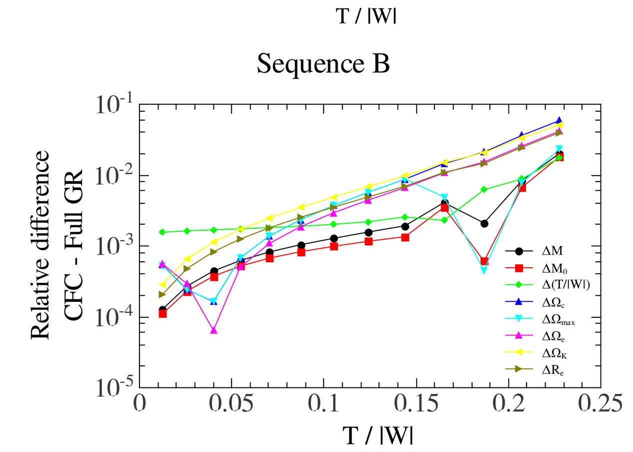


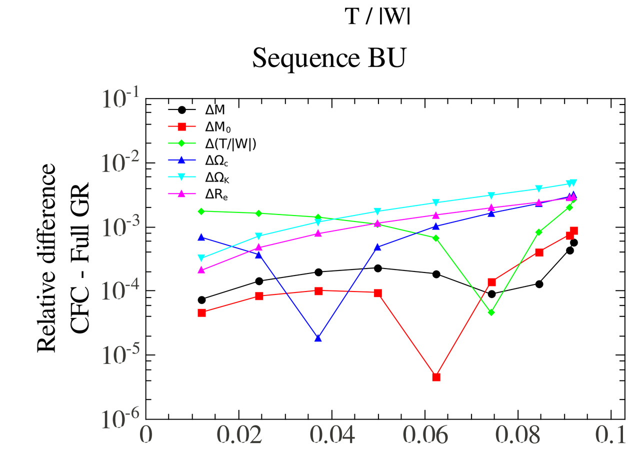












Conclusions

The CFC approximation appears to be a robust method to study systems that exhibit differential rotation if the demands for accuracy are not particularly strict, i.e if one can cope with a maximum error of around 5%. In most cases the errors encountered are *significantly* lower and one should examine the fastest rotating models to observe the maximum error mentioned above. In sequence A, all the relative differences calculated are below 1% and Δc for the fastest rotating model of the sequence remains below 2%. In sequence B larger deviations are observed, however only the relative differences for the angular velocities approach the maximum value of 6%. The diagnostic Δc for the fastest rotating model of sequence B indicates that the maximum deviation from full GR is only around 6%. This result should provide added confidence in choosing the CFC approximation as a possible candidate to tackle an astrophysically relevant problem that involves differential rotation.

As far as the case of uniform rotation is concerned, we verified that the CFC approximation works particularly well. For every physical quantity that was evaluated, the corresponding relative difference between CFC and full GR never exceeded 1%, staying mainly in the 10^{-4} to 10^{-3} range. In addition, the diagnostic Δc for the fastest uniformly rotating models remained under 2%.

Future directions for this project include the calculation of the Bach tensor for every configuration in order to add another diagnostic in our study of the CFC approximation. If the CFC approximation is a valid method for studying systems that rotate differentially, then the Bach tensor should vanish or otherwise be close to zero. We expect that the calculation of the Bach tensor will not alter significantly the already produced results but will further strengthen CFC as a satisfactory approximation of full GR.

References

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For further information

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