Can effects of quantum gravity be observed in the cosmic microwave background?

Claus Kiefer

Institut für Theoretische Physik
Universität zu Köln

Contents

Erwin Schrödinger 1926:

We know today, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures. Perhaps this failure is in strict analogy with the failure of geometrical optics . . . that becomes evident as soon as the obstacles or apertures are no longer great compared with the real, finite, wavelength. . . . Then it becomes a question of searching for an undulatory mechanics, and the most obvious way is by an elaboration of the Hamiltonian analogy on the lines of undulatory optics.[1]

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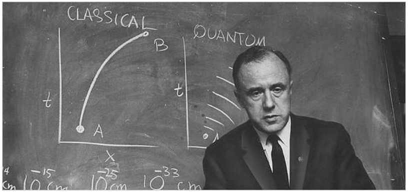
[1] *wir wissen doch heute, daß unsere klassische Mechanik bei sehr kleinen Bahndimensionen und sehr starken Bahnkrümmungen versagt*. Vielleicht ist dieses Versagen eine volle Analogie zum Versagen der geometrischen Optik . . . , das bekanntlich eintritt, sobald die 'Hindernisse' oder 'Öffnungen' nicht mehr groß sind gegen die wirkliche, endliche Wellenlänge. . . . Dann gilt es, eine 'undulatorische Mechanik' zu suchen – und der nächstliegende Weg dazu ist wohl die wellentheoretische Ausgestaltung des Hamiltonschen Bildes.

Conservative route to quantum gravity

'Quantization':

- The mechanical Hamilton–Jacobi equation leads to the Schrödinger equation;
- the gravitational Hamilton–Jacobi equation (Peres 1962) leads to the Wheeler–DeWitt equation.

Quantum geometrodynamics



- Question: what is the quantum wave equation that immediately gives Einstein's equations in the semiclassical limit?
- Answer: the Wheeler–DeWitt equation

$$\boxed{\hat{H}\Psi = 0}$$

Constraints of this type also occur in loop quantum gravity

Problem of time

- External time t has disappeared from the formalism
- This also holds for loop quantum gravity and probably for string theory
- Ψ depends on the three-dimensional metric $h_{ab}(\mathbf{x})$ and all non-gravitational degrees of freedom
- Wheeler–DeWitt equation has the structure of a hyperbolic equation and thus allows the introduction of an intrinsic time: for a Friedmann model with perturbations, the hyperbolicity holds with respect to the scale factor
- Hilbert-space structure in quantum mechanics is connected with the probability interpretation, in particular with probability conservation *in time* t; what happens with this structure in a timeless situation?

Semiclassical (Born–Oppenheimer) approximation

Ansatz:

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]\mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S[h_{ab}]}|\psi[h_{ab}]\rangle$$

and expansion with respect to the Planck-mass squared.

Highest order: One evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical Einstein equations, $h_{ab}(\mathbf{x}, t)$, corresponding to a solution, $S[h_{ab}]$, of the Hamilton–Jacobi equations;

$$\dot{h}_{ab} = NG_{abcd}\frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}$$

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle := \int \mathrm{d}^3x\, \dot{h}_{ab}(\mathbf{x},t)\, \frac{\delta}{\delta h_{ab}(\mathbf{x})}|\psi[h_{ab}]\rangle$$

This leads to a functional Schrödinger equation for quantized matter fields in the chosen external classical gravitational field:

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle &= \hat{H}^{\mathrm{m}}|\psi(t)\rangle \\ \hat{H}^{\mathrm{m}} &:= \int \mathrm{d}^3x \left\{ N(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{\perp}(\mathbf{x}) + N^a(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_a(\mathbf{x}) \right\} \end{aligned}$$

$\hat{H}^{\mathrm{m}}$: matter-field Hamiltonian in the Schrödinger picture, parametrically depending on (generally non-static) metric coefficients of the curved space–time background.

WKB time t controls the dynamics in this approximation

Quantum gravitational corrections

The next order in the Born–Oppenheimer approximation gives

$$\hat{H}^{\mathrm{m}} \to \hat{H}^{\mathrm{m}} + \frac{1}{m_{\mathrm{P}}^2} \times (\text{various terms})$$

(C. K. and T. P. Singh (1991); A. O. Barvinsky and C. K. (1998))

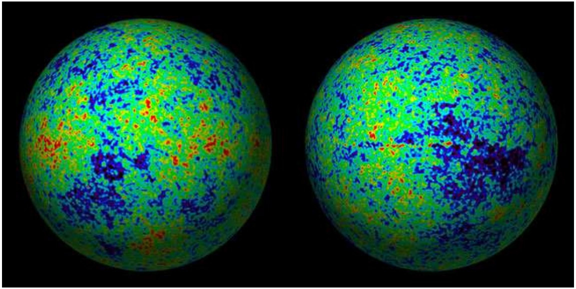
Example: Quantum gravitational correction to the trace anomaly in de Sitter space:

$$\delta\epsilon \approx -\frac{2G\hbar^2 H_{\mathrm{dS}}^6}{3(1440)^2\pi^3 c^8}$$

(C.K. 1996)

Observations

Does the anisotropy spectrum of the cosmic background radiation contain information about quantum gravity?



C.K. and M. Krämer, *Phys. Rev. Lett.*, **108**, 021301 (2012); see also our essay awarded first prize in the Gravity Research Foundation competition 2012.

Minisuperspace background

Wheeler–DeWitt equation for small fluctuations in a flat Friedmann–Lemaître universe with scale factor $a \equiv \exp(\alpha)$ and inflaton field ϕ

Choose the simplest potential:

$$\mathcal{V}(\phi) = \frac{1}{2}\, m^2 \phi^2 \ ;$$

any other potential obeying at the classical level the slow-roll condition $\dot{\phi}^2 \ll |\mathcal{V}(\phi)|$ should lead to similar results.

Minisuperspace Wheeler–DeWitt equation

$$\mathcal{H}_0\Psi_0(\alpha,\phi) \equiv \frac{\mathrm{e}^{-3\alpha}}{2}\left[\frac{1}{m_{\mathrm{P}}^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + \mathrm{e}^{6\alpha}m^2\phi^2\right]\Psi_0(\alpha,\phi) = 0$$

- $\hbar = c = 1$
- $m_{\mathrm{P}} = \sqrt{3\pi/2G} \approx 2.65 \times 10^{19}$ GeV
- $\phi \to \phi/\sqrt{2}\pi$
- assume in the following $\partial^2\Psi_0/\partial\phi^2 \ll \mathrm{e}^{6\alpha}m^2\phi^2\Psi_0$ and substitute $m\phi$ by $m_{\mathrm{P}}H$, where H is the quasistatic Hubble parameter of inflation (Born–Oppenheimer approximation)

Introduction of inhomogeneities

$$\phi \to \phi(t) + \delta\phi(\mathbf{x}, t)$$

Perform a decomposition into Fourier modes with wave vector $\mathbf{k}$, $k \equiv |\mathbf{k}|$,

$$\delta\phi(\mathbf{x}, t) = \sum_k f_k(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,.$$

The Wheeler-DeWitt equation including the fluctuation modes then reads (Halliwell and Hawking 1985)

$$\left[\mathcal{H}_0 + \sum_{k=1}^{\infty} \mathcal{H}_k\right] \Psi\big(\alpha, \phi, \{f_k\}_{k=1}^{\infty}\big) = 0$$

with

$$\mathcal{H}_k = \frac{1}{2}\,\mathrm{e}^{-3\alpha}\left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2\,\mathrm{e}^{4\alpha} + m^2\,\mathrm{e}^{6\alpha}\right) f_k^2\right]$$

$$\mathcal{H}_k = \frac{1}{2}\,\mathrm{e}^{-3\alpha}\left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2\,\mathrm{e}^{4\alpha} + m^2\,\mathrm{e}^{6\alpha}\right) f_k^2\right]$$

Ansatz:

$$\Psi\left(\alpha, \phi, \{f_k\}_{k=1}^{\infty}\right) = \Psi_0(\alpha, \phi) \prod_{k=1}^{\infty} \widetilde{\Psi}_k(\alpha, \phi, f_k)\,.$$

The components $\Psi_k(\alpha, \phi, f_k) := \Psi_0(\alpha, \phi)\widetilde{\Psi}_k(\alpha, \phi, f_k)$ obey

$$\frac{1}{2}\,\mathrm{e}^{-3\alpha}\left[\frac{1}{m_{\mathrm{P}}^2}\,\frac{\partial^2}{\partial\alpha^2} + \mathrm{e}^{6\alpha}\,m_{\mathrm{P}}^2\,H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha) f_k^2\right]\Psi_k(\alpha, \phi, f_k) = 0$$

with

$$W_k(\alpha) := k^2\,\mathrm{e}^{4\alpha} + m^2\,\mathrm{e}^{6\alpha}\,,$$

Born–Oppenheimer approximation

Following the general scheme, we make the ansatz

$$\Psi_k(\alpha, f_k) = \mathrm{e}^{\mathrm{i}\, S(\alpha, f_k)}$$

and expand $S(\alpha, f_k)$ in terms of powers of m_{P}^2,

$$S(\alpha, f_k) = m_{\mathrm{P}}^2\, S_0 + m_{\mathrm{P}}^0\, S_1 + m_{\mathrm{P}}^{-2}\, S_2 + \ldots$$

We insert this ansatz into the full Wheeler–DeWitt equation and compare consecutive orders of m_{P}^2.

- $\mathcal{O}(m_{\rm P}^4)$: S_0 is independent of f_k
- $\mathcal{O}(m_{\rm P}^2)$: S_0 obeys the Hamilton–Jacobi equation

$$\left[\frac{\partial S_0}{\partial \alpha}\right]^2 - V(\alpha) = 0\,, \quad V(\alpha) := \mathrm{e}^{6\alpha} H^2$$

  solved by $S_0(\alpha) = \pm \mathrm{e}^{3\alpha} H/3$
- $\mathcal{O}(m_{\rm P}^0)$: Write $\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha)\,\mathrm{e}^{\mathrm{i}\,S_1(\alpha, f_k)}$ and impose a condition on $\gamma(\alpha)$ that makes it equal to the standard WKB prefactor. After introducing the 'WKB time' according to

$$\frac{\partial}{\partial t} := -\,\mathrm{e}^{-3\alpha} \frac{\partial S_0}{\partial \alpha}\,\frac{\partial}{\partial \alpha}\,,$$

  one finds that each $\psi_k^{(0)}$ obeys a Schrödinger equation,

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(0)} = \mathcal{H}_k \psi_k^{(0)}.$$

Quantum gravitational corrections

$\mathcal{O}(m_{\mathrm{P}}^{-2})$: decompose $S_2(\alpha, f_k)$ as

$$S_2(\alpha, f_k) \equiv \varsigma(\alpha) + \eta(\alpha, f_k)$$

and demand that $\varsigma(\alpha)$ be the standard second-order WKB correction. The wave functions

$$\psi_k^{(1)}(\alpha, f_k) := \psi_k^{(0)}(\alpha, f_k)\, \mathrm{e}^{\mathrm{i}\, m_{\mathrm{P}}^{-2}\, \eta(\alpha, f_k)}$$

then obey the quantum gravitationally corrected Schrödinger equation

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(1)} = \mathcal{H}_k \psi_k^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2m_{\mathrm{P}}^2 \psi_k^{(0)}} \left[\frac{\left(\mathcal{H}_k\right)^2}{V} \psi_k^{(0)} + \mathrm{i}\,\frac{\partial}{\partial t} \left(\frac{\mathcal{H}_k}{V} \right) \psi_k^{(0)} \right] \psi_k^{(1)}$$

Solution of the uncorrected Schrödinger equation

Ansatz:

$$\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t)\, \mathrm{e}^{-\frac{1}{2}\,\Omega_k^{(0)}(t)\, f_k^2}$$

This leads to

$$\begin{aligned}
\dot{\mathcal{N}}_k^{(0)}(t) &= -\frac{\mathrm{i}}{2}\, \mathrm{e}^{-3\alpha}\, \mathcal{N}_k^{(0)}(t)\, \Omega_k^{(0)}(t), \\
\dot{\Omega}_k^{(0)}(t) &= \mathrm{i}\, \mathrm{e}^{-3\alpha} \Big[-\big(\Omega_k^{(0)}(t)\big)^2 + W_k(t) \Big].
\end{aligned}$$

Solution:

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2 \xi}\, \frac{1}{\xi - \mathrm{i}} + \mathcal{O}\left(\frac{m^2}{H^2}\right)$$

$$\xi(t) := k/(H a(t))$$

Unperturbed power spectrum

In the slow-roll regime, the density contrast is given by

$$\delta_k(t) \approx \frac{\delta\rho_k(t)}{\mathcal{V}_0} = \frac{\dot{\phi}(t)\,\dot{\sigma}_k(t)}{\mathcal{V}_0}\,,$$

with

$$\sigma_k^2(t) := \left\langle \psi_k | f_k^2 | \psi_k \right\rangle = \sqrt{\frac{\Re\mathfrak{e}\,\Omega_k}{\pi}} \int\limits_{-\infty}^{\infty} f_k^2\, \mathrm{e}^{-\frac{1}{2}[\Omega_k^*(t)+\Omega_k(t)]f_k^2}\, \mathsf{d}f_k = \frac{1}{2\,\Re\mathfrak{e}\,\Omega_k(t)}$$

$$\delta_k(t_{\text{enter}}) = \frac{4}{3}\,\frac{\mathcal{V}_0}{\dot{\phi}^2}\,\delta_k(t_{\text{exit}}) = \frac{4}{3}\,\frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)}\bigg|_{t\,=\,t_{\text{exit}}}$$

$$\Delta^2_{(0)}(k) := 4\pi k^3 \left|\delta_k(t_{\text{enter}})\right|^2 \propto \frac{H^4}{\left|\dot{\phi}(t)\right|^2_{t_{\text{exit}}}}$$

(approximately scale-invariant power spectrum)

Solution of the corrected Schrödinger equation

Ansatz:

$$\begin{aligned}\psi_k^{(1)}(t,f_k) &= \left(\mathcal{N}_k^{(0)}(t)+\frac{1}{m_\mathrm{P}^2}\,\mathcal{N}_k^{(1)}(t)\right)\\ &\times \exp\left[-\,\frac{1}{2}\left(\Omega_k^{(0)}(t)+\frac{1}{m_\mathrm{P}^2}\,\Omega_k^{(1)}(t)\right)f_k^2\right]\end{aligned}$$

Inserting this into the corrected Schrödinger equation leads to

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\,\Omega_k^{(1)}(\xi)=\frac{2\,\mathrm{i}\,\xi}{\xi-\mathrm{i}}\,\Omega_k^{(1)}(\xi)+\frac{3\,\xi^3}{2}\,\frac{2\xi-\mathrm{i}}{(\xi-\mathrm{i})^3}\;,$$

which can be solved analytically up to a numerical integration.

Modification of the power spectrum

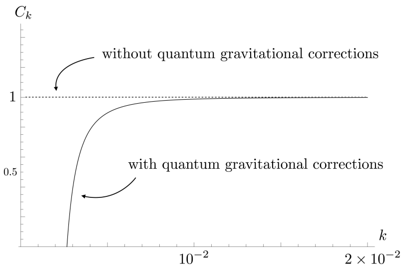


Figure: Size of the corrections for $H = 10^{14}$ GeV.

$$\Delta^2_{(1)}(k) = \Delta^2_{(0)}(k)\, C_k^2$$

$$C_k := \left(1 - \frac{43.56}{k^3}\,\frac{H^2}{m_{\mathsf{P}}^2}\right)^{-\frac{3}{2}}\left(1 - \frac{189.18}{k^3}\,\frac{H^2}{m_{\mathsf{P}}^2}\right)$$

introduces scale dependence

Discussion

- Effect is most pronounced for large scales
- Accuracy is fundamentally limit by cosmic variance
- Suppression of power for large scales
- From the non-observation of this effect, one finds the bound
$$H \lesssim 1.4 \times 10^{-2}\, m_{\mathsf{P}} \sim 4 \times 10^{17}\, \text{GeV}$$
- But there already exists a stronger constraint on this scale from the bound on the tensor-to-scalar ratio r :
$$H \lesssim 10^{-5} m_{\mathsf{P}} \sim 10^{14}\, \text{GeV}$$

Comparison with loop quantum cosmology

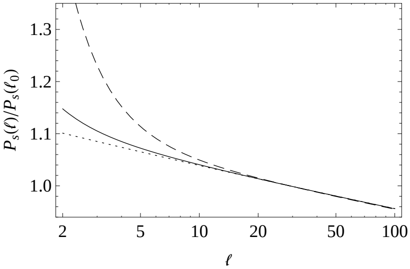


Figure: Primordial power spectrum for a certain model of loop quantum cosmology (upper curve). The dotted line is the classical case, and the solid line is the experimental upper bound. From: M. Bojowald, G. Calcagni, and S. Tsujikawa, *Phys. Rev. Lett.*, **107**, 211302 (2011).

Loop quantum cosmology predicts an *enhancement* of power at large scales.

Summary

- Concrete prediction from a conservative approach to quantum gravity
- It is consistent with existing observational limits
- No additional trans-Planckian effects are needed to understand these predictions
- In the present case, the effect is too small to be observable, but maybe one can find testable predictions along these lines
- Comparison with other approaches: loop quantum cosmology predicts an enhancement at large scales, while other approaches (non-commutative geometry, string-inspired cosmology) seem to predict a suppression