Gravitational lensing by a Kerr-(anti) de Sitter black hole.

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Abstract

1 Introduction .

The gravitational bending of light (and the associate phenomenon of gravitational lensing) has been instrumental in unravelling the nature of the gravitational field and its cosmological implications.

- – Despite the importance of the gravitational bending of light not many exact analytic results for the *deflection angle* from the gravitational field of important astrophysical objects are known in the literature.
  – One of the most important physical objects for Astrophysics and Gravitational Astronomy is the Kerr black hole.
  – The full analytic treatment of the Kerr and Kerr-de Sitter black holes as *gravitational lenses* was imperative since the Kerr black hole acts as a very strong gravitational lens and we may probe General Relativity, through the phenomenon of the bending of light induced by the spacetime curvature of a spinning black hole, at the *strong gravitational field regime*.
  – The cosmological constant Λ is the prime culprit responcible for the force that dominates the current observed expansion of the Universe and triggered the onset of the accelerated expansion shortly before the formation of the solar system. It had been debated as to whether or not the cosmological constant bends light. Indeed, as we shall see today Λ does contribute to the gravitational bending of light.
  – In this talk I am happy to report my recent results in obtaining the full analytic solution for the deflection angle of generic light orbits in Kerr-de Sitter black hole spacetime, as well as the analytic treatment of the more involved issue of treating the Kerr and Kerr-de Sitter black holes as gravitational lenses G. V. Kraniotis, Clas.Quantum Grav. 28(2011) 085021.
  – I will also report new results concerning the propagation of the polarization vector in Kerr spacetime. These results utilize the exact analytic solutions of the lens equations in combination with the application of the Walker-Penrose constant in order to study the change of the polarization vector and the associated rotation of the polarization plane (gravitational Faraday effect) at the strong field regime.

2 Null geodesics in Kerr and Kerr-de Sitter spacetime.

Taking into account the contribution from the cosmological constant, Λ, the generalization of the Kerr solution is described by the Kerr-de Sitter metric element which in Boyer-Lindquist (BL) coordinates is given by [Stuchlik Calvani,Gen.R.Grav23 (1991),Demianski(1973)ActaAstron.23, Carter,Com.M.P.(1968)]:

$$\begin{aligned} \mathrm{d}s^2 = & \frac{\Delta_r}{\Xi^2\rho^2}(c\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi)^2 - \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\ & - \frac{\Delta_\theta \sin^2\theta}{\Xi^2\rho^2}(ac\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2 \end{aligned} \tag{1}$$

$$\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\theta,\ \Xi := 1 + \frac{a^2\Lambda}{3},\ \rho^2 = r^2 + a^2\cos^2\theta, \tag{2}$$

$$\Delta_r := \left(1 - \frac{\Lambda}{3}r^2\right)\left(r^2+a^2\right) - 2\frac{GM}{c^2}r \tag{3}$$

(rdel)

We denote by a the rotation (Kerr) parameter and M denotes the mass of the spinning black hole.

The relevant null geodesic differential equations for the calculation of the gravitational lensing effects (lens-equation) and for the calculation of the deflection angle are:

Kraniotis,CQG22(2005)4391–4424

$$\boxed{\int^r \frac{\mathrm{d}r}{\sqrt{R}} = \pm \int^\theta \frac{\mathrm{d}\theta}{\sqrt{\Theta}}} \tag{4}$$

and Tr2

$$\boxed{\Delta\phi = \int \mathrm{d}\phi = \int^\theta -\frac{\Xi^2}{\Delta_\theta \sin^2\theta}\frac{(a\sin^2\theta - \Phi)\mathrm{d}\theta}{\sqrt{\Theta}} + \int^r \frac{a\Xi^2}{\Delta_r}[(r^2+a^2) - a\Phi]\frac{\mathrm{d}r}{\sqrt{R}}} \tag{5}$$

(lens),(ray)

where

$$R := \left\{ \Xi^2 \left[(r^2 + a^2) - a\Phi \right]^2 - \Delta_r \left[\Xi^2 \left(\Phi - a \right)^2 + \mathcal{Q} \right] \right\} \tag{6}$$

and

$$\Theta := \left\{ [\mathcal{Q} + (\Phi - a)^2 \Xi^2] \Delta_\theta - \frac{\Xi^2 (a \sin^2 \theta - \Phi)^2}{\sin^2 \theta} \right\} \tag{7}$$

We also derive the equation related to time-delay:

$$ct = \int^r \frac{\Xi^2 (r^2 + a^2) \left[(r^2 + a^2) - \Phi a \right]}{\pm \Delta_r \sqrt{R}} \mathrm{d}r - \int^\theta \frac{a \Xi^2 (a \sin^2 \theta - \Phi)}{\pm \Delta_\theta \sqrt{\Theta}} \mathrm{d}\theta. \tag{8}$$

The parameters $\Phi, \mathcal{Q}$ are associated to the first integrals of motion. The former is the impact parameter and the latter is related to the hidden first integral (due to the separation of variables in the corresponding Hamilton-Jacobi PDE).

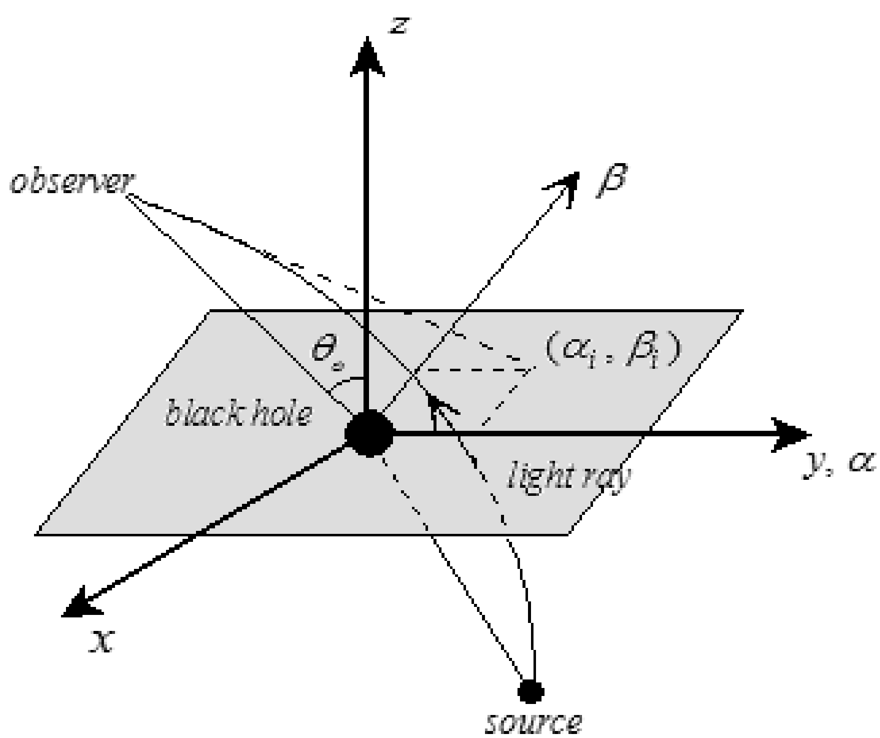
We now turn our attention to the issue of treating the Kerr and Kerr-de Sitter black holes as gravitational lenses, construct the resulting geometry and lens equations and solve the latter in closed analytic form. In addition, we shall derive for the first time the solutions in closed analytic form for the magnification factors.

Previous efforts on the issue of gravitational lensing from a Kerr black hole were concentrated on various approximations and numerical techniques Bray I., *Phys.Rev.*D. **34** (1986) 367;S.E. Vazquez and E.P. Esteban, Nuov.Cim. 119B(2004) 489, M. Sereno and F. De Luca, *Phys.Rev.*D. **74** (2006) 123009, arXiv:astro-ph/0609435v2; V. Bozza, F. De Luca and G. Scarpetta, *Phys.Rev.*D. **74** (2006) 063001; V. Bozza, *Phys.Rev.*D. **78** (2008) 063014.

3 The Kerr black hole as a gravitational lens

Assume without loss of generality that the observer's position is at $(r_O, \theta_O, 0)$. Likewise, for the source we have (r_S, θ_S, ϕ_S) . In the observer's reference frame, an incoming light ray is described by a parametric curve $x(r), y(r), z(r)$, where $r^2 = x^2 + y^2 + z^2$. For large r this the usual radial BL coordinate.

At the location of the observer, the tangent vector to the parametric curve is given by: $(\mathrm{d}x/\mathrm{d}r)|_{r_O} \widehat{\mathbf{x}} + (\mathrm{d}y/\mathrm{d}r)|_{r_O} \widehat{\mathbf{y}} + (\mathrm{d}z/\mathrm{d}r)|_{r_O} \widehat{\mathbf{z}}$. This vector describes a straight line which intersects the (α, β) plane or *observer's image plane* as it is usually called at (α_i, β_i). The point (α_i, β_i) is the point $(-\beta_i \cos\theta_O, \alpha_i, \beta_i \cos\theta_O)$ in the (x, y, z) system.


z
observer
β
θ₀
(α_i , β_i)
black hole
y, α
light ray
x
source

Our purpose now is to relate the α_i, β_i variables to the first integrals of motion $\Phi, \mathcal{Q}$. For this we need to use the equation of straight line in space. A straight line can be defined from a point $P_1(x_1, y_1, z_1)$ on it and a vector $\overline{\epsilon}(\epsilon_1,\epsilon_2,\epsilon_3)$ parallel to it. The analytic equations of straight line are then:

$$\frac{x-x_1}{\epsilon_1}=\frac{y-y_1}{\epsilon_2}=\frac{z-z_1}{\epsilon_3} \tag{9}$$

Applying (9) we derive the equations:

$$\frac{-\beta_i\cos\theta_O - r_O\sin\theta_O}{r_O\cos\theta_O\frac{d\theta}{dr}|_{r=r_0}+\sin\theta_O}=\frac{\alpha_i}{r_O\sin\theta_O\frac{d\phi}{dr}|_{r=r_O}}=\frac{\beta_i\cos\theta_O - r_O\cos\theta_O}{\cos\theta_O - r_O\sin\theta_O\frac{d\theta}{dr}|_{r=r_O}} \tag{10}$$

Solving for α_i, β_i we obtain the equations:

$$\alpha_i = -r_O^2\sin\theta_O\frac{d\phi}{dr}|_{r=r_O} \tag{11}$$

$$\beta_i = r_O^2\frac{d\theta}{dr}|_{r=r_O} \tag{12}$$

Now we have from the null geodesics that:

$$\frac{d\theta}{dr}|_{r=r_O}=\frac{\Theta(\theta_O)^{1/2}}{R(r_O)^{1/2}} \tag{13}$$

and

$$\frac{d\phi}{dr}|_{r=r_O}=\frac{\Phi}{\sqrt[2]{R(r_O)}}\frac{1}{\sin^2(\theta_O)}+\frac{2aGM\frac{r_O}{c^2}-a^2\Phi}{r_O^2\left[1+\frac{a^2}{r_O^2}-\frac{2GM}{r_Oc^2}\right]}\frac{1}{\sqrt[2]{R(r_O)}} \tag{14}$$

Using eqns(13),(14) and assuming large observer's distance r_O(i.e. $r_O \longrightarrow \infty$) we derive simplified expressions relating the coordinates (α_i, β_i) on the observer's image plane to the integrals of motion:

$$\Phi \simeq -\alpha_i\sin\theta_O \tag{15}$$

$$\mathcal{Q} \simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2(\theta_O) \tag{16}$$

(IP)

We can also express the position of the source on the observer's sky in terms of its coordinates (r_S, θ_S, ϕ_S) and the observer coordinates. Indeed, the equation for a straight line can be determined by two points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \tag{17}$$

Thus applying the above formula for the straight line connecting the observer and the source yields the equations:

$$\alpha_S=\frac{r_O r_S \sin\theta_S \sin\phi_S}{r_O-r_S(\cos\theta_S\cos\theta_O+\sin\theta_O\sin\theta_S\cos\phi_S)} \tag{18}$$

$$\beta_S=\frac{-r_O r_S(\sin\theta_O\cos\theta_S-\sin\theta_S\cos\phi_S\cos\theta_O)}{r_O-r_S(\cos\theta_S\cos\theta_O+\sin\theta_O\sin\theta_S\cos\phi_S)} \tag{19}$$

4 Magnification factors and positions of images.

The flux of an image of an infinitesimal source is the product of its surface brightness and the solid angle $\Delta\omega$ it subtends on the sky. Since the former quantity is unchanged during light deflection, the ratio of the flux of a sufficiently small image to that of its corresponding source in the absence of the lens, is given by

$$\mu=\frac{\Delta\omega}{(\Delta\omega)_0}=\frac{1}{|J|} \tag{20}$$

where 0-subscripts denote undeflected quantities and J is the Jacobian of the transformation $(x_S, y_S)\rightarrow(x_i,y_i)$ [1]. Writting $x_S=x_S(x_i,y_i), y_S=y_S(x_i,y_i)$ we can find expressions for the partial derivatives appearing in the Jacobian by differentiating equations (4) and (5). Indeed, the Jacobian is given by the expression:

$$J=xw-zy \tag{21}$$

where we defined: $x:=\frac{\partial x_S}{\partial x_i}, y:=\frac{\partial x_S}{\partial y_i}, z:=\frac{\partial y_S}{\partial x_i}, w:=\frac{\partial y_S}{\partial y_i}$. Writting equations (4) and (5) as follows(pvsol2) :

$$\begin{aligned} R_1(x_i,y_i)-A_1(x_i,y_i,x_S,y_S,m)=0 \\ \Delta\phi(x_S,y_S,n)-R_2(x_i,y_i)-A_2(x_i,y_i,x_S,y_S,m)=0 \end{aligned} \tag{22}$$

[1] Recall in the small angles approxiamation: $\alpha_i\approx r_O\ x_i$, $\beta_i\approx r_O\ y_i$. Also we define: $x_S:=\frac{\alpha_S}{r_O}, y_S:=\frac{\beta_S}{r_O}$.

(wpf) (Aij)we set up the following system of equations:

$$\beta_1 = -\alpha_1 x - \alpha_2 z \tag{23}$$
$$\beta_2 = -\alpha_1 y - \alpha_2 w \tag{24}$$
$$-\beta_3 = \alpha_3 x + \alpha_4 z \tag{25}$$
$$-\beta_4 = \alpha_3 y + \alpha_4 w \tag{26}$$

where $\alpha_1 = \frac{\partial A_1}{\partial x_S},\ \alpha_2 = \frac{\partial A_1}{\partial y_S},\ \alpha_3 = -\frac{\partial \phi_s}{\partial x_S} - \frac{\partial A_2}{\partial x_S},\ \alpha_4 = -\frac{\partial \phi_s}{\partial y_S} - \frac{\partial A_2}{\partial y_S}$,

$\beta_1 = \frac{\partial R_1}{\partial x_i} - \frac{\partial A_1}{\partial x_i},\ \beta_2 = \frac{\partial R_1}{\partial y_i} - \frac{\partial A_1}{\partial y_i},\ \beta_3 = \frac{\partial R_2}{\partial x_i} + \frac{\partial A_2}{\partial x_i},\ \beta_4 = \frac{\partial R_2}{\partial y_i} + \frac{\partial A_2}{\partial y_i}$.

Solving for x, y, z, w we obtain:

(Magnification)

$$\mu = \frac{1}{|J|} = \left| \frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{\beta_1\beta_4 - \beta_2\beta_3} \right| \tag{27}$$

5 The boundary of the shadow of the Kerr black hole.

The condition for a photon to escape to infinity , which is also the condition for the spherical photon orbits in Kerr spacetime, is given by the vanishing of the quartic polynomial $R(r)$ and its first derivative (also in this case $\frac{d^2R}{dr^2}|_{r=r_f} > 0$). Implementing these two conditions, expressions for the parameter Φ and Carter's constant $\mathcal{Q}$ are obtained:

$$\Phi = \frac{a^2\frac{GM}{c^2} + a^2 r - 3\frac{GM}{c^2}r^2 + r^3}{a\left(\frac{GM}{c^2} - r\right)}, \qquad \mathcal{Q} = -\frac{r^3\left(-4a^2\frac{GM}{c^2} + r\left(\frac{-3GM}{c^2} + r\right)^2\right)}{a^2\left(\frac{GM}{c^2} - r\right)^2},$$

The perturbed, from the radius $r = r_{\rm inst}$ of unstable spherical null orbits in Kerr spacetime, and thus escaped photon, will be detected on the observer's image plane, at the coordinates:

$$x_i = \frac{a^2(r + \frac{GM}{c^2}) + r^2(r - \frac{3GM}{c^2})}{r_O \sin\theta_O a\left(r - \frac{GM}{c^2}\right)},$$
$$y_i = \frac{\pm\sqrt{-r^3[r(r - \frac{3GM}{c^2})^2 - 4a^2\frac{GM}{c^2}] - 2a^2 r(2a^2\frac{GM}{c^2} + r^3 - 3r\frac{G^2M^2}{c^4})z_O - a^4(r - \frac{GM}{c^2})^2 z_O^2}}{r_O \sin\theta_O a\left(r - \frac{GM}{c^2}\right)} \tag{28}$$

A photon will be detected when the argument of the square root in 28 is positive. Also, $z_O := \cos^2\theta_O$. In Kraniotis, CQG 28 (2011) 085021 various constraints for the motion of light from the allowed polar region: $\theta_{\min} \leq \theta_S, \theta_O \leq \theta_{\max}$ were derived ($z_m \geq z_O$ etc).

6 Closed form solution for the angular integrals.

In this case we have to take into account the turning points in the polar coordinate. A generic angular polar integral can be written:

$$\pm \int_{\theta_1}^{\theta_2} = \int_{\min(z_1,z_2)}^{\max(z_1,z_2)} + [1 - \mathrm{sign}(\theta_1 \circ \theta_2)] \int_0^{\min(z_1,z_2)} \tag{29}$$

where:

$$\theta_1 \circ \theta_2 := \cos\theta_1 \cos\theta_2 \tag{30}$$

Indeed, using the variable $z := \cos^2\theta$ we derive:

$$-\frac{1}{2}\frac{\mathrm{d}z}{\sqrt{z}}\frac{1}{\sqrt{1-z}} = \mathrm{sign}(\frac{\pi}{2} - \theta)\mathrm{d}\theta \tag{31}$$

This is the result of the fact that in the interval $0 \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$ and $\sin\theta \geq 0$, while in the interval $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin\theta \geq 0$, $\cos\theta \leq 0$.

Now, for a light trajectory that encounters m turning points ($m \geq 1$) we have:

$$\int^{\theta} = \pm \int_{\theta_S}^{\theta_{\min/\max}} \underbrace{\pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} \pm \int_{\theta_{\max/\min}}^{\theta_{\min/\max}} \cdots}_{m-1 \text{ times}} \pm \int_{\theta_{\max/\min}}^{\theta_O} = \tag{32}$$

$$\begin{aligned} &= \int_{z_S}^{z_m} + [1 - \mathrm{sign}(\theta_S \circ \theta_{mS})] \int_0^{z_S} \\ &\quad + \int_{z_O}^{z_m} + [1 - \mathrm{sign}(\theta_O \circ \theta_{mO})] \int_0^{z_O} \\ &\quad + 2(m-1) \int_0^{z_m} \end{aligned} \tag{33}$$

The roots z_m, z_3 (of $\Theta(\theta) = 0$) are expressed in terms of the integrals of motion and the cosmological constant by the expressions:

$$z_{m,3} = \frac{\mathcal{Q} + \Phi^2\Xi^2 - H^2 \pm \sqrt{(\mathcal{Q} + \Phi^2\Xi^2 - H^2)^2 + 4H^2\mathcal{Q}}}{-2H^2} \tag{34}$$

and(Lk)

$$H^2 := \frac{a^2\Lambda}{3}[\mathcal{Q} + (\Phi - a)^2\Xi^2] + a^2\Xi^2 \tag{35}$$

For $\Lambda = 0$, the turning points take the form:

$$\boxed{z_m = \frac{a^2 - \mathcal{Q} - \Phi^2 + \sqrt{4a^2\mathcal{Q} + (-a^2 + \mathcal{Q} + \Phi^2)^2}}{2a^2},} \tag{36}$$

where the subscript "m" stands for "min/max". The corresponding angles are:

$$\boxed{\theta_{\text{min/max}} = \text{Arccos}(\pm\sqrt{z_m})} \tag{37}$$

.

Also:

$$\boxed{\theta_{mO} := \text{Arccos}(\text{sign}(y_i)\sqrt{z_m}),} \tag{38}$$

and

$$\theta_{mS} := \begin{cases} \theta_{mO}, & m \,\text{odd} \\ \pi - \theta_{mO}, & m \,\text{even} \end{cases}. \tag{39}$$

(mt)

Now for θ_j and $\theta_{\text{min / max}}$ in the same hemisphere:

$$\int_{\theta_j}^{\theta_{\text{min / max}}} \frac{\mathrm{d}\theta}{\pm\sqrt[2]{\Theta(\theta)}} = \frac{1}{2|a|}\int_{z_j}^{z_m} \frac{\mathrm{d}z}{\sqrt[2]{z(z_m - z)(z - z_3)}} \equiv I_3 \tag{40}$$

Let us now calculate the elliptic integral in eqn.(40) in closed analytic form. Applying the transformation:

$$z = z_m + \xi^2(z_j - z_m) \tag{41}$$

our integral is calculated in closed form in terms of Appell's generalized hypergeometric function F_1 of two variables:

$$I_3 = \frac{1}{2|a|}\frac{\sqrt[2]{(z_m - z_j)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_j}{z_m}, \frac{z_m - z_j}{z_m - z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \tag{42}$$

The function $F_1(\alpha,\beta,\beta',\gamma,x,y)$ is the first of the four Appell's hypergeometric functions of two variables x,y (Appell,1882, J.Math.Pures Appl.Liouville 8,173-216),

$$\boxed{F_1(\alpha,\beta,\beta',\gamma,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}x^m y^n} \tag{43}$$

which admits the following integral representation:

$$\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y) \tag{44}$$

The double series converges when:

$$\boxed{|x|<1,\quad |y|<1} \tag{45}$$

The above *Euler* integral representation is valid for: $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\gamma-\alpha)>0$. *Also* $\Gamma(p)=\int_0^{\infty}x^{p-1}e^{-x}dx$ denotes the Gamma function. The Pochhammer symbol $(\alpha)_m=(\alpha,m)$ is defined as:

$$(\alpha)_m=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}=\begin{cases}1, & \text{if} \quad m=0\\ \alpha(\alpha+1)\cdots(\alpha+m-1) & \text{if } m=1,2,3\end{cases} \tag{46}$$

On the other hand using the transformation:

$$\boxed{z=\frac{uz_jz_m-z_jz_m}{uz_j-z_m}} \tag{47}$$

we calculate in closed form:

$$\begin{aligned}&\frac{1}{2|a|}\int_0^{z_j}\frac{\mathrm{d}z}{\sqrt[2]{z(z_m-z)(z-z_3)}}\\ &=\frac{1}{|a|}\frac{\sqrt[2]{z_j}}{z_m}\sqrt[2]{\frac{z_j-z_m}{z_3-z_j}}F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j}{z_m},\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}\right)\\ &=\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}}}{\sqrt[2]{z_m-z_3}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)},\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}\right)\end{aligned} \tag{48}$$

In going from the second line to the third of (48) we made use of the following identity of Appell's first generalised hypergeometric function of two variables:

$$F_1(\alpha,\beta,\beta',\gamma,x,y)=(1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}F_1(\gamma-\alpha,\beta,\gamma-\beta-\beta',\gamma,\frac{x-y}{x-1},y) \tag{49}$$

Likewise we derive the closed form solution for the following integral:

$$\begin{aligned}
&\frac{1}{2|a|}\int_0^{z_j}\frac{\mathrm{d}z}{(1-z)\sqrt[2]{z(z_m-z)(z-z_3)}}\\
&=\frac{z_j}{z_m}\frac{1}{|a|}\frac{z_j-z_m}{1-z_j}\frac{1}{\sqrt[2]{z_j(z_j-z_m)(z_3-z_j)}}\times\\
&\quad F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j(1-z_m)}{z_m(1-z_j)},\frac{z_j}{z_m},\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}\right)\\
&=\frac{1}{|a|}\frac{z_j}{z_m}\sqrt[2]{\frac{z_m}{-z_3z_j}}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},z_j,\frac{z_j}{z_m},\frac{z_j}{z_3}\right)
\end{aligned} \tag{50}$$

Producing the last line of equation (50) we used the following formula for the Lauricella function F_D (FD):

Proposition 1

$$\begin{aligned}
F_D(\alpha,\beta,\beta',\beta'',\gamma,x,y,z)=&(1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''}\times\\
&F_D\left(\gamma-\alpha,\beta,\gamma-\beta-\beta'-\beta'',\beta'',\gamma,\frac{x-y}{x-1},y,\frac{z-y}{z-1}\right)
\end{aligned}$$

Proof. Applying the transformation:

$$u=\frac{1-\nu}{1-\nu y} \tag{51}$$

onto the integral:

$$IR_{F_D}=\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-u\,y)^{-\beta'}(1-uz)^{-\beta''}\mathrm{d}u \tag{52}$$

we derive:

$$\begin{aligned}
&(1-u)^{\gamma-\alpha-1}=\left(\frac{\nu(1-y)}{1-\nu y}\right)^{\gamma-\alpha-1},\quad (1-ux)^{-\beta}=\left(\frac{(1-x)[1-\frac{\nu(x-y)}{(x-1)}]}{1-\nu y}\right)^{-\beta}\\
&(1-u\,y)^{-\beta'}=\frac{(1-y)^{-\beta'}}{(1-\nu y)^{-\beta'}},\quad (1-uz)^{-\beta''}=\left(\frac{(1-z)[1-\frac{\nu(z-y)}{z-1}]}{1-\nu y}\right)^{-\beta''}
\end{aligned} \tag{53}$$

and thus we obtain the result:

$$IR_{F_D} = (1-y)^{\gamma-\alpha}(1-x)^{-\beta}(1-y)^{-\beta'}(1-z)^{-\beta''} \times \int_0^1 \mathrm{d}\nu\, \nu^{\gamma-\alpha-1}(1-\nu)^{\alpha-1}(1-\nu\, y)^{-(\gamma-\beta-\beta'-\beta'')}(1-\nu\frac{x-y}{x-1})^{-\beta}(1-\nu\frac{z-y}{z-1})^{-\beta''} \tag{54}$$

or

$$F_D(\alpha,\beta,\beta',\beta'',\gamma,x,y,z) = (1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''} \times F_D\left(\gamma-\alpha,\beta,\gamma-\beta-\beta'-\beta'',\beta'',\gamma,\frac{x-y}{x-1},y,\frac{z-y}{z-1}\right)$$

■

Likewise:

$$I_4 := \frac{-\Phi}{2|a|}\int_{z_j}^{z_m}\frac{\mathrm{d}z}{(1-z)\sqrt[2]{z(z_m-z)(z-z_3)}} = \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_j)}{z_m}}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j-z_m}{1-z_m},\frac{z_m-z_j}{z_m},\frac{z_m-z_j}{z_m-z_3}\right) \tag{55}$$

Let us see for instance the term in (39):

$$\pm\int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2\int_0^{z_m} \tag{56}$$

since $\cos^2\theta_{\min/\max} = z_m$ and $\theta_{\min}\circ\theta_{\max} = -z_m$.(angul.)

Equation (55) for $z_j = 0$, becomes (tpL):

$$\begin{aligned} &-\frac{\Phi}{2|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{-z_m}{1-z_m},1,\frac{z_m}{z_m-z_3}\right)\\ &=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{1}{(1-z_m)}\frac{\pi}{2}F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{-z_m}{1-z_m},\frac{z_m}{z_m-z_3}\right)\\ &=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{\pi}{2}F_1\left(\frac{1}{2},1,-\frac{1}{2},1,\frac{z_m(1-z_3)}{z_m-z_3},\frac{z_m}{z_m-z_3}\right)\\ &=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{\pi}{2}\frac{1}{1-z_3}\left(F(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3})-z_3F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{z_m(1-z_3)}{z_m-z_3},\frac{z_m}{z_m-z_3}\right)\right) \end{aligned} \tag{57}$$

On the other hand the angular integrals of the form $\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}}$ in equation (5) are solved in closed analytic form as follows:

$$\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}} = \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)$$
$$+[1-\text{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\frac{z_S}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt[2]{z_S(z_S-z_m)(z_3-z_S)}}\times$$
$$F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_{3)}}\right) \quad (58)$$

(isim)An equivalent expression for the above integral is(F):

$$\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}} = \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_S-z_m}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)$$
$$+[1-\text{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\sqrt[2]{\frac{z_S}{z_m}}\sqrt[2]{\frac{z_m-z_3}{z_S-z_3}}\frac{1}{\sqrt[2]{z_m-z_3}}\times$$
$$F_D\left(\frac{1}{2},1,\frac{1}{2},-\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)$$
$$= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_S-z_m}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)$$
$$+[1-\text{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\sqrt[2]{\frac{z_S}{z_m}}\sqrt[2]{\frac{z_m-z_3}{z_S-z_3}}\frac{1}{\sqrt[2]{z_m-z_3}}\times$$
$$[\frac{-z_3}{1-z_3}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)+$$
$$\frac{1}{1-z_3}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)] \quad (59)$$

Thus we have that (2len):

$$A_2(x_i, y_i, x_S, y_S, m) = 2(m-1) \times \left[-\frac{\Phi}{|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{1}{(1-z_m)} \frac{\pi}{2} F_1\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{-z_m}{1-z_m}, \frac{z_m}{z_m - z_3}\right)\right]$$
$$+ \frac{-\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_S)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1-z_m)} \times$$
$$F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S - z_m}{1-z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right)$$
$$+ [1 - \mathrm{sign}(\theta_S \circ \theta_{ms})](-) \frac{\Phi}{|a|} \frac{z_S}{z_m} \frac{z_S - z_m}{1 - z_S} \frac{1}{\sqrt[2]{z_S(z_S - z_m)(z_3 - z_S)}} \times$$
$$F_D\left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}\right) +$$
$$+ \frac{-\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_O)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1-z_m)} \times$$
$$F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_O - z_m}{1-z_m}, \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3}\right)$$
$$[1 - \mathrm{sign}(\theta_O \circ \theta_{mO})](-) \frac{\Phi}{|a|} \frac{z_O}{z_m} \frac{z_O - z_m}{1 - z_O} \frac{1}{\sqrt[2]{z_O(z_O - z_m)(z_3 - z_O)}} \times$$
$$F_D\left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_O(1-z_m)}{z_m(1-z_O)}, \frac{z_O}{z_m}, \frac{z_O(z_m - z_3)}{z_m(z_O - z_3)}\right) \tag{60}$$

Lauricella's 4^{th} hypergeometric function of m-variables.

$$F_D(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{z}) = \sum_{n_1, n_2, \ldots, n_m = 0}^{\infty} \frac{(\alpha)_{n_1 + \cdots n_m} (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}}{(\gamma)_{n_1 + \cdots + n_m} (1)_{n_1} \cdots (1)_{n_m}} z_1^{n_1} \cdots z_m^{n_m} \tag{61}$$

(tr1)

where

$$\mathbf{z} = (z_1, \ldots, z_m),$$
$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m). \tag{62}$$

The Pochhammer symbol $(\alpha)_m = (\alpha, m)$ is defined by

$$(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } \quad m = 0 \\ \alpha(\alpha+1)\cdots(\alpha+m-1) & \text{if } m = 1, 2, 3 \end{cases} \tag{63}$$

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With the notations $\mathbf{z}^{\mathbf{n}} := z_1^{n_1} \cdots z_m^{n_m}$, $(\boldsymbol{\beta})_{\mathbf{n}} := (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}$, $\mathbf{n}! = n_1! \cdots n_m!$, $|\mathbf{n}| := n_1 + \cdots + n_m$ for m-tuples of numbers in (62) and of non-negative integers $\mathbf{n} = (n_1, \cdots n_m)$ the Lauricella series F_D in compact form is:

$$F_D(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{z}) := \sum_{\mathbf{n}} \frac{(\alpha)_{|\mathbf{n}|}(\boldsymbol{\beta})_{\mathbf{n}}}{(\gamma)_{|\mathbf{n}|}\mathbf{n}!}\mathbf{z}^{\mathbf{n}} \tag{64}$$

The series admits the following integral representation:

$$F_D(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{z}) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z_1 t)^{-\beta_1} \cdots (1-z_m t)^{-\beta_m} \mathrm{d}t \tag{65}$$

which is valid for $\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\gamma - \alpha) > 0.$. It converges absolutely inside the m-dimensional cuboid:

$$|z_j| < 1, (j = 1, \ldots, m). \tag{66}$$

For $m = 2$, F_D in the notation of Appell becomes the two variable hypergeometric function $F_1(\alpha, \beta, \beta', \gamma, x, y)$.

For purposes of presentation, we define the following tuples of numbers for the beta parameters and the variables of the function F_D that will occur in our closed form solutions:

$$
\begin{aligned}
&\mathbf{z_j^1} = \left(\frac{z_j - z_m}{1-z_m}, \frac{z_m - z_j}{z_m}, \frac{z_m - z_j}{z_m - z_3}\right), \;\; \mathbf{j} = 1,2, \mathbf{z_1^1} \equiv \mathbf{z_S^1}, \mathbf{z_2^1} \equiv \mathbf{z_O^1}, \\
&\mathbf{z_S^1} = \left(\frac{z_S - z_m}{1-z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right), \;\; \mathbf{z_S^2} = \left(\frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}\right), \\
&\mathbf{z_O^1} = \left(\frac{z_O - z_m}{1-z_m}, \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3}\right), \;\; \mathbf{z_O^2} = \left(\frac{z_O(1-z_m)}{z_m(1-z_O)}, \frac{z_O}{z_m}, \frac{z_O(z_m - z_3)}{z_m(z_O - z_3)}\right), \\
&\mathbf{z_S^3} = \left(\frac{z_S(1-z_3)}{z_S - z_3}, \frac{z_S}{z_m}\frac{(z_m - z_3)}{(z_S - z_3)}, \frac{z_S}{z_S - z_3}\right), \\
&\boldsymbol{\beta_3^1} = \left(2, \frac{1}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_3^2} = \left(1, \frac{3}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_3^3} = \left(1, \frac{1}{2}, \frac{3}{2}\right), \;\; \boldsymbol{\beta_3^4} = \left(1, \frac{1}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_3^5} = \left(2, \frac{-1}{2}, \frac{1}{2}\right), \\
&\boldsymbol{\beta_3^6} = \left(1, \frac{-1}{2}, \frac{3}{2}\right), \;\; \boldsymbol{\beta_3^7} = \left(1, \frac{-1}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_3^8} = \left(1, \frac{1}{2}, -\frac{1}{2}\right), \; \boldsymbol{\beta_3^9} = \left(\frac{1}{2}, 1, \frac{1}{2}\right), \\
&\boldsymbol{\beta_4^{10}} = \left(-2, 2, \frac{1}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_4^{11}} = \left(-1, 1, \frac{1}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_4^{\Lambda 1}} = \left(1, 1, \frac{1}{2}, \frac{1}{2}\right), \\
&\boldsymbol{\beta_4^{\Lambda 2}} = \left(1, 1, -\frac{3}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_4^{\Lambda 3}} = \left(-1, \frac{1}{2}, \frac{1}{2}, 1\right)
\end{aligned} \tag{67}
$$

and the corresponding 2-tuples for the two-variable Appell's first hypergeometric function F_1:

$$
\begin{aligned}
&\mathbf{z_A^{1S}} = \left(\frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right), \;\; \mathbf{z_A^{1O}} = \left(\frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3}\right), \;\; \mathbf{z_{AaO}^{td}} = \left(\frac{z_O}{z_m}, \frac{z_O}{z_m}\frac{z_m - z_3}{z_O - z_3}\right), \\
&\mathbf{z_A^{2S}} = \left(\frac{z_m}{z_m - z_3}\frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}\right), \;\; \mathbf{z_A^{2O}} = \left(\frac{z_m}{z_m - z_3}\frac{z_O(z_m - z_3)}{z_m(z_O - z_3)}, \frac{z_O(z_m - z_3)}{z_m(z_O - z_3)}\right), \\
&\mathbf{z_{AaS}^{td}} = \left(\frac{z_S}{z_m}, \frac{z_S}{z_m}\frac{z_m - z_3}{z_S - z_3}\right), \;\; \mathbf{z_{AO_2}^{1/4}} = \left(\frac{z_m}{z_m - z_3}\frac{(1/4)(z_m - z_3)}{z_m((1/4) - z_3)}, \frac{(1/4)(z_m - z_3)}{z_m((1/4) - z_3)}\right), \\
&\mathbf{z_{AO_1}^{1/4}} = \left(\frac{z_m - \frac{1}{4}}{z_m}, \frac{z_m - \frac{1}{4}}{z_m - z_3}\right), \;\; \boldsymbol{\beta_A^{ra}} = \left(\frac{1}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_A^{a1}} = \left(\frac{3}{2}, \frac{1}{2}\right), \;\; \boldsymbol{\beta_A^{a2}} = \left(\frac{1}{2}, \frac{3}{2}\right), \;\; \boldsymbol{\beta_A^{a3}} = \left(-\frac{1}{2}, \frac{1}{2}\right)
\end{aligned} \tag{68}
$$

The angular integrals of the form $\pm\int_{\theta_S}^{\theta_{\min/\max}}$ in equation (4) are calculated in closed-analytic form as follows:

$$\pm\int_{\theta_S}^{\theta_{\min/\max}}\frac{d\theta}{\sqrt[2]{\Theta}}=\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}$$
$$+[1-\mathrm{sign}(\theta_s\circ\theta_{ms})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times$$
$$F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right) \tag{69}$$

7 Closed form solution for the lens equations.

We now present our closed form solution for the Kerr black hole. We begin the presentation of the solution from the angular terms appearing in 22 Kraniotis,CQG 28 (2011) 085021:

Theorem 2

$$A_2(x_i,y_i,x_S,y_S,m)=2(m-1)\times\left[\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{1}{(1-z_m)}\frac{\pi}{2}F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{-z_m}{1-z_m},\frac{z_m}{z_m-z_3}\right)\right]$$
$$+\frac{\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\times F_D\left(\frac{1}{2},\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{4}},\frac{3}{2},\mathbf{z}_{\mathbf{S}}^{\mathbf{1}}\right)$$
$$+[1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{\Phi}{|a|}\frac{z_S}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt[2]{z_S(z_S-z_m)(z_3-z_S)}}\times$$
$$F_D\left(1,\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{7}},\frac{3}{2},\mathbf{z}_{\mathbf{S}}^{\mathbf{2}}\right)+$$
$$+\frac{\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_O)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\times F_D\left(\frac{1}{2},\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{4}},\frac{3}{2},\mathbf{z}_{\mathbf{O}}^{\mathbf{1}}\right)$$
$$+[1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{\Phi}{|a|}\frac{z_O}{z_m}\frac{z_O-z_m}{1-z_O}\frac{1}{\sqrt[2]{z_O(z_O-z_m)(z_3-z_O)}}\times$$
$$F_D\left(1,\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{7}},\frac{3}{2},\mathbf{z}_{\mathbf{O}}^{\mathbf{2}}\right) \tag{70}$$

Theorem 3

$$A_1(x_i, y_i, x_S, y_S, m) = 2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right) + \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{1S}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} + [1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{2S}}\right) + \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_O)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{1O}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} + [1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}}}{\sqrt[2]{z_m-z_3}}\times F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{2O}}\right) \tag{71}$$

8 Closed form solution for the radial integrals.

We now perform the radial integration assuming $\Lambda = 0$:(AO)

For an observer and a source located far away from the black hole, the relevant radial integrals can take the form:

$$\int^{r} \rightarrow -\int_{r_S}^{\alpha} + \int_{\alpha}^{r_O} \simeq 2\int_{\alpha}^{\infty} \tag{72}$$

For instance we meet the radial integral:

$$\int_{\alpha}^{\infty}\frac{aE}{\Delta}[(r^2+a^2)-a\Phi]\frac{dr}{\sqrt[2]{R}} \tag{73}$$

In order to calculate the contribution to the deflection angle from the radial term we need to integrate the above equation from the distance of closest approach (e.g., from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic by $\alpha,\beta,\gamma,\delta : \alpha > \beta > \gamma > \delta$.

It suffices to proceed with the term:

$$\int_{\alpha}^{\infty}\frac{a\frac{2GM}{c^2}r - a^2\Phi}{\Delta\sqrt[2]{R}}\mathrm{d}r \tag{74}$$

In G. V. Kraniotis, Clas.Quantum.Grav. 28 (2011) 085021 the following theorem was proved:

Theorem 4 $2\int_{\alpha}^{\infty}\frac{a\frac{2GM}{c^2}r-a^2\Phi}{\Delta\sqrt[2]{R}}\mathrm{d}r$

$$
\begin{aligned}
&=2\left[\frac{-2A_{+}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{+}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{+}^{r}}\right)\right.\\
&\quad+\frac{A_{+}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{+}}\left(-\frac{1}{\kappa_{+}'^{2}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^r}\right)2\right.\\
&\quad\left.+\frac{1}{\kappa_{+}'^{2}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{+}^{r}}\right)2\right)\\
&\quad+\frac{-2A_{-}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{-}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{-}^{r}}\right)\\
&\quad+\frac{A_{-}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{-}}\left(-\frac{1}{\kappa_{-}'^{2}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^r}\right)2\right.\\
&\quad\left.\left.+\frac{1}{\kappa_{-}'^{2}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{-}^{r}}\right)2\right)\right]\\
&\equiv R_2(x_i,y_i) \qquad (75)
\end{aligned}
$$

R2 NHA

Theorem 5 *Also:* $\int_{\alpha}^{\infty}\frac{\mathrm{d}r}{\sqrt{R}}=\frac{1}{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\boldsymbol{\beta}_A^{\mathbf{ra}},\frac{3}{2},\mathbf{z_A^r}\right).$

We exploit further the lens equations:

$$
\begin{aligned}
&R_1(x_i,y_i)-2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)\\
&+\cdots=\int^{\xi_S}\frac{\mathrm{d}\xi}{\sqrt{4\xi^3-g_2\xi-g_3}}. \qquad (76)
\end{aligned}
$$

Inverting yields the Weierstraß modular function:

$$\xi_S=\wp(\mathrm{lhs}(76)+\epsilon). \qquad (77)$$

9 Summary of the solution for the lens equation

Let us summarize our closed form solution for the lens equations in Kerr black hole Kraniotis,CQG 28 (2011) 085021:

$$\xi_S = \wp\left(R_1(x_i,y_i) - 2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)+\cdots+\epsilon\right) \tag{78}$$

The balance equation :

$$\begin{aligned} R_1(x_i,y_i) \equiv 2\int_\alpha^\infty \frac{1}{\sqrt{R}}\mathrm{d}r = A_1(x_i,y_i,x_S,y_S,m) \Leftrightarrow & \frac{2}{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\boldsymbol{\beta}^{\boldsymbol{ra}}_{\boldsymbol{A}},\frac{3}{2},\mathbf{z}^{\mathbf{r}}_{\mathbf{A}}\right) \\ &= 2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right) \\ &+ \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta}^{\boldsymbol{ra}}_{\boldsymbol{A}},\frac{3}{2},\mathbf{z}^{\mathbf{1S}}_{\mathbf{A}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times F_1\left(\frac{1}{2},\boldsymbol{\beta}^{\boldsymbol{ra}}_{\boldsymbol{A}},\frac{3}{2},\mathbf{z}^{\mathbf{2S}}_{\mathbf{A}}\right) \\ &+ \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_O)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta}^{\boldsymbol{ra}}_{\boldsymbol{A}},\frac{3}{2},\mathbf{z}^{\mathbf{1O}}_{\mathbf{A}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}}}{\sqrt[2]{z_m-z_3}}\times F_1\left(\frac{1}{2},\boldsymbol{\beta}^{\boldsymbol{ra}}_{\boldsymbol{A}},\frac{3}{2},\mathbf{z}^{\mathbf{2O}}_{\mathbf{A}}\right), \end{aligned} \tag{79}$$

$$-\phi_S = R_2(x_i,y_i) + A_2(x_i,y_i,x_S,y_S,m) \tag{80}$$

theor1 retsol while the Weierstraß invariants are given in terms of the initial conditions by:

$$g_2 = \frac{1}{12}(\alpha+\beta)^2 - \mathcal{Q}\frac{\alpha}{4}, g_3 = \frac{1}{216}(\alpha+\beta)^3 - \mathcal{Q}\frac{\alpha^2}{48} - \mathcal{Q}\frac{\alpha\beta}{48}. \tag{81}$$

Also $\alpha := -a^2, \beta := \mathcal{Q}+\Phi^2$, $z_S = -\frac{\xi_S+\frac{\alpha+\beta}{12}}{-\alpha/4}$, and ϵ is a constant of integration.

10 Specific examples.

Positions of Images , source for an equatorial observer.

In this case ($\theta_O = \pi/2$), and the equations relating the first integrals of motion to the coordinates on the observer's image plane become:

$$\begin{aligned} \Phi &\simeq -\alpha_i \sin\theta_O = -\alpha_i \\ \mathcal{Q} &\simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2\theta_O = \beta_i^2. \end{aligned} \tag{82}$$

Thus the length of the vector on the observer's image plane equals:

$$\sqrt{\alpha_i^2 + \beta_i^2} = \sqrt{\Phi^2 + \mathcal{Q}}. \tag{83}$$

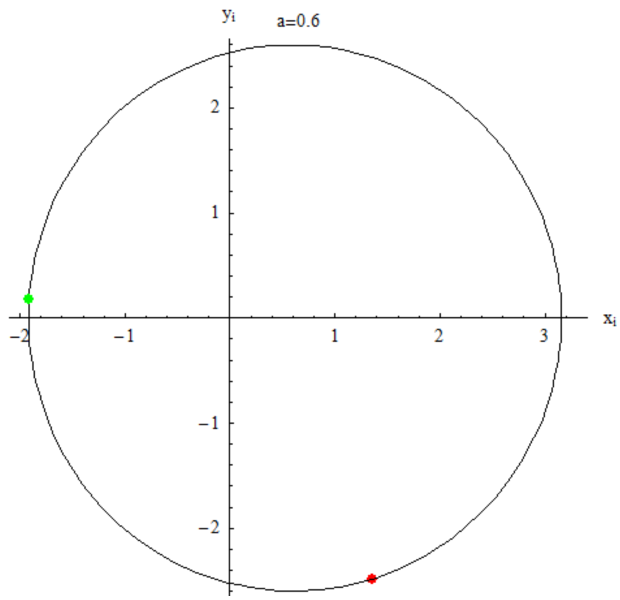
For a choice of initial conditions $a, \Phi, \mathcal{Q}$ we determine values for the observer's image plane coordinates α_i, β_i. Subsequently, we determine the value of z_S that solves the balance equation, exploiting the exact solution for z_S in terms of the Weierstraß elliptic function $\wp$. We must determine at which regions of the $fundamental\ period\ parallelogram$ the Weierstraß function takes real and negative values. Indeed:

$$\boxed{\wp(\frac{\omega}{l} + \omega'; g_2, g_3) \in \mathbb{R}^-, for\ x = \frac{\omega}{l} + \omega', \quad l \in \mathbb{R}} \tag{84}$$

Having determined θ_S we determine the azimuthal position of the source ϕ_S by (80)

	$a = 0.6, \mathcal{Q} = 24.64563, \Phi = -2.71910$	$a = 0.6, \mathcal{Q} = 0.128, \Phi = 3.839$
$\alpha_i\ (\frac{GM}{c^2})$	2.719110	−3.839
$\beta_i\ (\frac{GM}{c^2})$	−4.9644365239	0.357770876399
$x_i \left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	1.359555	−1.9195
$y_i(\frac{2}{r_O}\frac{GM}{c^2})$	−2.48221826	0.178885
m	3	3
z_S	0.3161007914992452	0.0026145818604
θ_S	55.79 °	87.069 °
$\Delta\phi$(rad)	−11.086	7.09441
ϕ_S	95.1794 °	133.52 °
ω	0.5545341990201503500	0.824718843878947
ω'	1.3278669366032567973i	2.9400828459149726i

Assuming that the galactic centre region SgrA*, is a Kerr black hole with mass: $M_{BH} = 4.06 \times 10^6\ M_\odot$ and a distance from the observer to the galactic centre: $r_O = 8$Kpc, the second solution (green image) will require an angular resolution of 19.3102μarcs. This is in the

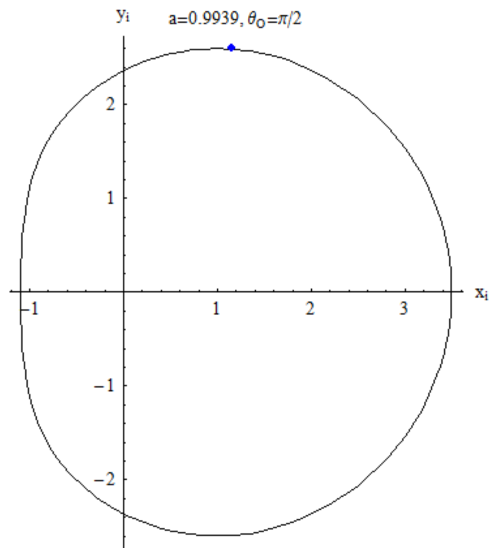


range of experimental accuracy for both the under construction TMT and GRAVITY experiments.

We repeat the analysis for a higher value for the spin of the black hole.

	$a = 0.9939$, $\mathcal{Q} = 27.0220588123$, $\Phi = -2.29885534$
α_i $\left(\frac{GM}{c^2}\right)$ β_i $\left(\frac{GM}{c^2}\right)$ x_i $\left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	2.29885534 5.198274599547431 1.14942767
$y_i\left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	2.5991372997737154
m	3
z_S	0.01378435185109
θ_S	83.2575°
$\Delta\phi$(rad)	−11.243
ϕ_S	104.177°
ω	0.5505433970950226
ω'	1.1288708298860726 i

We observe in this case, that the boundary of the shadow of the black hole is not perfectly circular as it tends to be for low value (in fact for $a = 0$) of the Kerr black hole. The observation of this shadow would be direct evidence of an **event horizon** and the departure from a perfect circle for the ring of light will provide experiment evidence for the spin of



the black hole. This will lead to a measurement of black hole's angular momentum.

11 Magnifications for an equatorial observer in a Kerr black hole.

In this case $(\theta_O = \pi/2)$, equations (15),(16), become:

$$\Phi \simeq -\alpha_i \sin\theta_O = -\alpha_i \tag{85}$$

$$\mathcal{Q} \simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2\theta_O = \beta_i^2 \tag{86}$$

and

$$x_S := \frac{\alpha_S}{r_O} = \frac{r_S \sin\theta_S \sin\phi_S}{r_O - r_S \sin\theta_S \cos\phi_S} \tag{87}$$

$$y_S := \frac{\beta_S}{r_O} = \frac{-r_S \cos\theta_S}{r_O - r_S \sin\theta_S \cos\phi_S} \tag{88}$$

$$
\begin{aligned}
&\frac{\partial(59)}{\partial x_S} = \frac{\partial(59)}{\partial z_S}\frac{\partial z_S}{\partial x_S},\\
&\frac{\partial(59)}{\partial z_S} = \frac{\Phi}{2|a|}\frac{1}{z_m}\frac{1}{(1-z_m)}\frac{1}{\sqrt{z_m-z_3}}\left(\frac{z_m-z_S}{z_m}\right)^{-1/2}\times\\
&\quad F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+\\
&\quad\left(\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\right)\times\Bigg\{\\
&\quad F_D\left(\frac{3}{2},2,\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{1}{1-z_m}+\\
&\quad F_D\left(\frac{3}{2},1,\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{-1}{z_m}+\\
&\quad F_D\left(\frac{3}{2},1,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{-1}{z_m-z_3}\Bigg\}+\\
&\quad(1-\mathrm{sign}(\theta_S\circ\theta_{ms})\,(-1)\Bigg[\Big[\frac{1}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}+\\
&\quad\frac{z_S}{z_m}\frac{z_3(z_m-3z_Sz_m+2z_S^2)-z_S(z_m(2-4z_S)+z_S(-1+3z_S))}{2(1-z_S)^2z_S(z_3-z_S)\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\Big]\times\\
&\quad F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+\\
&\quad\frac{z_S}{z_m}\frac{z_S-z_m}{(1-z_S)}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\Bigg\{\\
&\quad F_D\left(2,2,-\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\frac{1-z_m}{z_m(1-z_S)^2}+\\
&\quad F_D\left(2,1,\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\frac{1}{z_m}+\\
&\quad F_D\left(2,1,\frac{-1}{2},\frac{3}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{-z_3(z_m-z_3)}{z_m(z_S-z_3)^2}\right)\Bigg\}\Bigg]
\end{aligned}
\tag{89}
$$

Now we calculate the term: $\frac{\partial(??)}{\partial z_S}$. Indeed, calculating the derivatives w.r.t. z_S we derive the expression:

$$\frac{\partial(\mathbf{??})}{\partial z_S}=\frac{1}{2|a|}\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)}\left(-\frac{1}{2\sqrt{z_m(z_m-z_3)}\sqrt{z_m-z_S}}\right)F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+$$
$$\frac{1}{2|a|}\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)}\sqrt{\frac{(z_m-z_S)}{z_m(z_m-z_3)}}\times\left[F_1\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\left(\frac{-1}{z_m}\right)+\right.$$
$$\left.F_1\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\left(\frac{-1}{z_m-z_3}\right)\right]+$$
$$[1-\mathrm{sign}(\theta_S\circ\theta_{ms})]\left[\frac{1}{2|a|}\left(\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)^{-\frac{1}{2}}\left\{\frac{(-z_3)\sqrt{z_m-z_3}}{z_m(z_S-z_3)^2}\right\}\times\right.$$
$$F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+$$
$$\frac{1}{|a|}\frac{\sqrt{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt{z_m-z_3}}\times\left[F_1\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{-z_3}{(z_S-z_3)^2}\right)+\right.$$
$$\left.\left.F_1\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{(-z_3)(z_m-z_3)}{z_m(z_S-z_3)^2}\right)\right]\right] \tag{90}$$

$$\alpha_1=\frac{\partial A_1}{\partial x_S}=(90)\times\frac{\partial z_S}{\partial x_S}=(90)\times\left(-2\cos\theta_S\sin\theta_S\times\frac{\frac{r_S^2\sin\theta_S\cos\theta_S\sin\phi_S}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right) \tag{91}$$

$$\alpha_2=\frac{\partial A_1}{\partial y_S}=(90)\times\frac{\partial z_S}{\partial y_S}=(90)\times\left(-2\cos\theta_S\sin\theta_S\times\frac{\frac{-[r_Or_S\sin\theta_S\cos\phi_S-r_S^2\sin^2\theta_S]}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right) \tag{92}$$

While for the α_3,α_4 terms which contrbute to the expression for the magnification, equation (27), m we derive the expressions:

$$\alpha_3=-\frac{\partial\phi_S}{\partial x_S}-\frac{\partial A_2}{\partial x_S}=-\left(-\frac{\frac{(r_Or_S\sin\theta_S-r_S^2\cos\phi_S)}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right)-(89)\times\left(\frac{\frac{r_S^2\sin\theta_S\cos\theta_S\sin\phi_S}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right) \tag{93}$$

,

$$\alpha_4 = -\frac{\partial \phi_S}{\partial y_S} - \frac{\partial A_2}{\partial y_S} = -\frac{\frac{r_O r_S \cos\theta_S \sin\phi_S}{(r_O - r_S \sin\theta_S \cos\phi_S)^2}}{J_1} - (89) \times \frac{\frac{-[r_O r_S \sin\theta_S \cos\phi_S - r_S^2 \sin^2\theta_S]}{(r_O - r_S \sin\theta_S \cos\phi_S)^2}}{J_1} \Bigg) \tag{94}$$

where J_1 denotes the Jacobian:

$$J_1 = \frac{\partial(x_S, y_S)}{\partial(\theta_S, \phi_S)} \tag{95}$$

and

$$\begin{aligned}
\frac{\partial \theta_S}{\partial x_S} &= \frac{(r_S^2 \sin\theta_S \cos\theta_S \sin\phi_S)/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial \theta_S}{\partial y_S} &= \frac{-[r_O r_S \sin\theta_S \cos\phi_S - r_S^2 \sin^2\theta_S]/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial \phi_S}{\partial x_S} &= -\frac{(r_O r_S \sin\theta_S - r_S^2 \cos\phi_S)/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial \phi_S}{\partial y_S} &= \frac{r_O r_S \cos\theta_S \sin\phi_S/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1}
\end{aligned} \tag{96}$$

In producing the results exhibited in eqns (89),(90) in our calculations for the magnification factors we made use of the important identity of Appell's hypergeometric function F_1 and its corresponding generalization for the Lauricella hypergeometric function F_D :

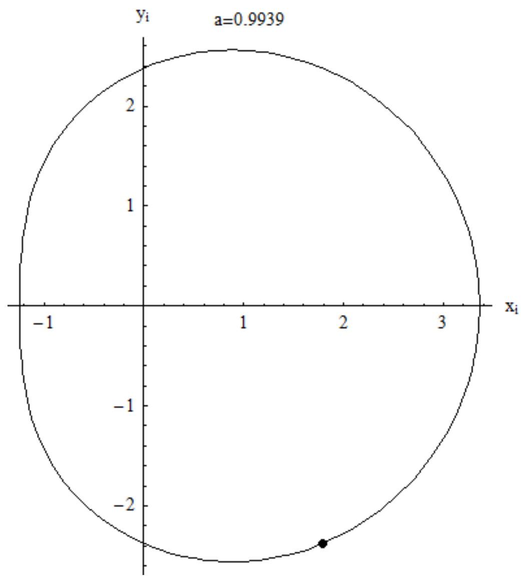
$$\frac{\partial^{m+n} F_1(\alpha, \beta, \beta', \gamma, x, y)}{\partial x^m \partial x^n} = \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \times F_1(\alpha+m+n, \beta+m, \beta'+n, \gamma+m+n, x, y) \tag{97}$$

Similar calculations that we do not exhibit in this talk leads to the derivation of the coefficients β_i.

12 Source and image positions for an observer located at $\theta_O = \frac{\pi}{3}$.

In this case:

$$\Phi = -\alpha_i \frac{\sqrt{3}}{2}, \qquad \mathcal{Q} = \beta_i^2 + \left(\frac{4\Phi^2}{3} - a^2\right)\frac{1}{4}. \tag{98}$$



	$a = 0.9939, \mathcal{Q} = 25.64563, \Phi = -3.11$	$a = 0.52, \mathcal{Q} = 23.64563, \Phi = -2.85$
$\alpha_i \left(\frac{GM}{c^2}\right)$	3.591118674	3.29089
$\beta_i \left(\frac{GM}{c^2}\right)$	−4.7611506980	−4.58320084657
$x_i \left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	1.79556	1.64545
$y_i \left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	−2.38058	−2.2916004
m	3	3
z_S	0.09097820848	0.5980171072414
θ_S	72.4447°	39.3474°
$\Delta\phi$(rad)	−12.0971	−11.8577
ϕ_S	153.112°	139.395°
ω	0.52792338858688228	0.5571026427501503
ω'	1.119903617249492i	1.389041935594241i

13 Exact solution of the angular integrals in the presence of Λ.

There has been a discussion in the literature as to whether or not the cosmological constant contributes to the gravitational lensing. However, the debate has been restricted to the Schwarzschild-de Sitter spacetime Lake (2007), Sereno,Phys.Rev.D 77(2008), Rindler,Phys.Rev.D76(2007). Let us discuss now the more general case of gravitational lensing in the Kerr-de Sitter spacetime.

The generalized solution for the angular integral (58) in the presence of Λ is given by:

$$\pm\int_{\theta_S}^{\theta_{\min/\max}} = \frac{\Xi^2}{2|H|}\frac{z_m - z_S}{(1-\eta z_m)}\frac{1}{\sqrt[2]{z_m(z_m-z_S)(z_m-z_3)}} \times \Bigg\{ + \frac{\Phi}{(1-z_m)} F_D\left(\frac{1}{2}, \boldsymbol{\beta_4^{\Lambda 1}}, \frac{3}{2}, \boldsymbol{z_{\Lambda}^{a1}}\right)\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} - aF_D\left(\frac{1}{2}, \boldsymbol{\beta_3^{4}}, \frac{3}{2}, \boldsymbol{z_{\Lambda}^{a2}}\right)\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\Bigg\} + (1-\mathrm{sign}(\theta_S \circ \theta_{mS}))\left[\frac{\Xi^2}{|H|}\frac{z_S}{z_m}\frac{z_m - z_S}{1-\eta z_S}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}} \times \left\{-aF_D\left(1, \boldsymbol{\beta_3^{7}}, \frac{3}{2}, \boldsymbol{z_{\Lambda}^{a4}}\right) + \frac{\Phi}{1-z_S}F_D\left(1, \boldsymbol{\beta_4^{\Lambda 2}}, \frac{3}{2}, \boldsymbol{z_{\Lambda}^{a3}}\right)\right\}\right] \tag{99}$$

where (H):

$$\eta := -\frac{a^2\Lambda}{3}, \mu = \frac{z_S}{z_m}\frac{z_m - z_3}{z_S - z_3}, \lambda = \frac{z_S}{z_m}\left(\frac{1-\eta z_m}{1-\eta z_S}\right), \nu = \frac{z_S}{z_m}\left(\frac{1-z_m}{1-z_S}\right)$$ and

$$\boldsymbol{z_{\Lambda}^{a1}} := \left(\frac{\eta(z_S - z_m)}{1-\eta z_m}, \frac{z_S - z_m}{1-z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right),$$
$$\boldsymbol{z_{\Lambda}^{a2}} := \left(\frac{\eta(z_S - z_m)}{1-\eta z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right),$$
$$\boldsymbol{z_{\Lambda}^{a3}} := \left(\lambda, \nu, \frac{z_S}{z_m}, \mu\right), \quad \boldsymbol{z_{\Lambda}^{a4}} := \left(\lambda, \frac{z_S}{z_m}, \mu\right) \tag{100}$$

Also the integrals $\pm\int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2\int_0^{z_m}$ contribute the term:

$$2(m-1)\times\Bigg\{\frac{\Xi^2\Phi}{2|H|}\frac{-z_m}{(1-\eta z_m)(1-z_m)}\frac{1}{\sqrt{z_m^2(z_m-z_3)}} \times F_D\Big(\frac{1}{2}, \boldsymbol{\beta_4^{\Lambda 1}}, \frac{3}{2}, \boldsymbol{z_{\Lambda 0}^{a1}}\Big)2 + \frac{-\Xi^2 a}{2|H|}\frac{z_m}{\sqrt{z_m^2(z_m-z_3)}}\frac{1}{1-\eta z_m} \times F_D\Big(\frac{1}{2}, \boldsymbol{\beta_3^{4}}, \frac{3}{2}, \boldsymbol{z_{\Lambda 0}^{a2}}\Big)2\Bigg\} \tag{101}$$

where the tuples of numbers $\boldsymbol{z_{\Lambda 0}^{a1}}$, $\boldsymbol{z_{\Lambda 0}^{a2}}$ appearing in the previous equation are defined by setting $z_S = 0$ in the tuples of numbers $\boldsymbol{z_{\Lambda}^{a1}}$, $\boldsymbol{z_{\Lambda}^{a2}}$ respectively. Notice that for $\Lambda = 0$ this reduces to equation (57).

14 Closed-form solution for radial integrals in the presence of Λ.

Assume first $\Lambda > 0$. We need to calculate radial integrals of the form:

$$\int \frac{a\Xi^2}{\Delta_r}((r^2+a^2)-a\Phi))\frac{\mathrm{d}r}{\sqrt[2]{R}} \tag{102}$$

We use the technique of partial fractions from integral calculus:

$$\frac{a\Xi^2}{\Delta_r}((r^2+a^2)-a\Phi)) = \frac{A^1}{r-r_\Lambda^+}+\frac{A^2}{r-r_\Lambda^-}+\frac{A^3}{r-r_+}+\frac{A^4}{r-r_-} \tag{103}$$

where $r_\Lambda^+, r_\Lambda^-, r_+, r_-$ are the four real roots of Δ_r (Der).

For instance, for $r_O, r_S < r_\Lambda^+$ one of the integrals we need to calculate is:

$$\frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)}}\int_\alpha^{r_\Lambda^+/2}\frac{A^1\mathrm{d}r}{(r-r_\Lambda^+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} \tag{104}$$

Indeed, we compute in closed form

$$\int_\alpha^{r_\Lambda^+/2}\frac{A^1\mathrm{d}r}{(r-r_\Lambda^+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} = \frac{\rho_1}{\sqrt{\rho_1}}H_\Lambda^+\times F_D\left(\frac{1}{2},\boldsymbol{\beta_4^{\Lambda 3}},\frac{3}{2},\boldsymbol{z^r_{\Lambda^+}}\right)\frac{\Gamma(1/2)}{\Gamma(3/2)} \tag{105}$$

where

$$\begin{aligned}
\rho_1 &:= \frac{r_\Lambda^+-\beta}{r_\Lambda^+-\alpha}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\\
\boldsymbol{z^r_{\Lambda^+}} &:= \left(\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{\beta-\gamma}{\alpha-\gamma}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{\beta-\delta}{\alpha-\delta}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{r_\Lambda^+-\beta}{r_\Lambda^+-\alpha}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta}\right)\\
H_\Lambda^+ &:= \frac{\alpha-\beta}{|\beta-\alpha|}\frac{1}{r_{\Lambda^+}-\beta}\frac{1}{\sqrt{\omega(\gamma-\alpha)(\delta-\alpha)}}
\end{aligned} \tag{106}$$

Also the radial integral involved in the lhs in the ‘balance’ lens equation is computed exactly in terms of the hypergeometric function of Appell F_1:

$$\frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)}}\int_{\alpha}^{r_\Lambda^+/2}\frac{\mathrm{d}r}{\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}}$$
$$=\frac{\rho_1}{\sqrt{\mathcal{E}}}\frac{1}{\sqrt{\omega(\gamma-\alpha)(\delta-\alpha)}}\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\boldsymbol{\beta^{ra}_A},\frac{3}{2},\boldsymbol{z^r_{A\Lambda+}}\right) \tag{107}$$

where $\mathcal{E}:=\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)$, $\omega:=\frac{r_\Lambda^+-\alpha}{r_\Lambda^+-\beta}$, $\boldsymbol{z^r_{A\Lambda+}}=\left(\frac{\beta-\gamma}{\alpha-\gamma}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{\beta-\delta}{\alpha-\delta}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta}\right)$ and $\alpha,\beta,\gamma,\delta$ denote the roots of the quartic polynomial R in the presence of Λ eqn(6).

Likewise, the generalization of equation (81) is given by:

$$\xi_S=\wp(2\times(107+\cdots+\epsilon) \tag{108}$$

where the Weierstraß invariants take the form

$$\begin{aligned} g_2&=\frac{1}{12}(\alpha_\Lambda+\beta_\Lambda)^2-\mathcal{Q}\frac{\alpha_\Lambda}{4},\\ g_3&=\frac{1}{216}(\alpha_\Lambda+\beta_\Lambda)^3-\mathcal{Q}\frac{\alpha_\Lambda^2}{48}-\mathcal{Q}\frac{\alpha_\Lambda\beta_\Lambda}{48} \end{aligned} \tag{109}$$

and

$$\alpha_\Lambda:=-H^2,\quad \beta_\Lambda:=\mathcal{Q}+\Phi^2\Xi^2. \tag{110}$$

A complete phenomenological analysis of our exact solutions in the presence of the cosmological constant Λwill be a subject of a separate publication. Nevertheless, it is evident from the closed form solutions we derived in this work that the cosmological constant does contribute to the gravitational bending of light.

15 The Polarization vector.

The polarization four-vector f^i of a linearly polarized light ray, which is orthogonal to the direction of propagation, must be parallel transported along a null geodesic with its tangent vector u^i (Penrose). The Kerr geometry is of Petrov type D, and the complex quantity:

$$K_{WP}=(A+iB)(r-ia\cos\theta) \tag{111}$$

is conserved along a null geodesic (Walker, Penrose), where

$$\begin{aligned} A&=(u^tf^r-u^rf^t)+a\sin^2\theta(u^rf^\phi-u^\phi f^r),\\ B&=(r^2+a^2)\sin\theta(u^\phi f^\theta-u^\theta f^\phi)-a\sin\theta(u^tf^\theta-u^\theta f^t). \end{aligned} \tag{112}$$

We can set $f^t = 0$, and using the orthogonality condition $u^i f_i = 0$, to eliminate f^r . Then by equating the real and imaginary parts of K_{WP} at the source and the observer:

$$\gamma_O \hat{f}^\theta_O - \beta_O \hat{f}^\phi_O = \gamma_S \hat{f}^\theta_S - \beta_S \hat{f}^\phi_S, \tag{113}$$

$$-\beta_O \hat{f}^\theta_O - \gamma_O \hat{f}^\phi_O = -\beta_S \hat{f}^\theta_S - \gamma_S \hat{f}^\phi_S, \tag{114}$$

we obtain the matrix R transforming the θ, ϕ components of the polarization vector at the source into the ones at the observer as follows (Ishihara *et al* PRD 38 1988):

$$\begin{bmatrix} \hat{f}^\theta \\ \hat{f}^\phi \end{bmatrix}_O = R \begin{bmatrix} \hat{f}^\theta \\ \hat{f}^\phi \end{bmatrix}_S, \tag{115}$$

where

$$R = (1+x^2)^{-1/2} \begin{bmatrix} 1 & -x \\ -x & -1 \end{bmatrix}. \tag{116}$$

The parameter x is defined as:

$$x := \frac{\beta_S \gamma_O - \gamma_S \beta_O}{\gamma_S \gamma_O + \beta_S \beta_O} \tag{117}$$

with

$$\gamma = \frac{\Phi}{\sin\theta} - a \sin\theta \tag{118}$$

and it is valid

$$\beta_S^2 + \gamma_S^2 = \beta_O^2 + \gamma_O^2 = \mathcal{Q} + (\Phi - a)^2. \tag{119}$$

For a geodesic joining the source and observer the parameter x can be computed. For instance for the first strong-field solution we derived (for $a = 0.6, \mathcal{Q} = 24.64563, \Phi = -2.71910$), the parameter x in the matrix R is: $x = 0.0972779$. This is the first strong-field calculation for the change in the polarization vector in Kerr spacetime.

Similarly, for the high spin exact solution of the lens equations with parameters $a = 0.9939, \mathcal{Q} = 27.0220588123, \Phi = -2.29885534$ we obtain the value for the x parameter $x = 0.00175849$.

16 Rotation of the plane of Polarization (Gravitational Faraday Effect).

At the distant places from the black hole any linear polarization vector $\vec{f}$ can be expressed as

$$\vec{f} = f_\perp \vec{\eta} + f_\parallel \vec{h} \tag{120}$$

where $\vec{\eta}$ and $\vec{h}$ are three-dimensional unit vectors first introduced by *Ishihara et al*, the first normal to the orbital plane and $\vec{h} := \vec{n} \times \vec{k}/|\vec{k}|$. Here $\vec{k}$ is a three-dimensional propagation vector

$$\boxed{\vec{k} = (k^r, rk^\theta, r\sin\theta k^\phi).} \tag{121}$$

. The component $f_\perp$ perpendicular to and $f_\parallel$ projected onto the orbital plane are given by

$$\begin{bmatrix} f_\parallel \\ f_\perp \end{bmatrix} = N \begin{bmatrix} f^\theta \\ f^\phi \end{bmatrix}, \tag{122}$$

where N is a matrix of the form

$$N = \begin{bmatrix} h^\theta \; h^\phi \\ \eta^\theta \; \eta^\phi \end{bmatrix} \tag{123}$$

We are particularly interested in the rotation angle

$$\begin{bmatrix} f_\parallel \\ f_\perp \end{bmatrix}_O = N_O R N_S^{-1} \begin{bmatrix} f_\parallel \\ f_\perp \end{bmatrix}_S. \tag{124}$$

This implies that the rotation of the plane of polarization is determined by the matrix $N_O R N_S^{-1}$:

$$N_O R N_S^{-1} = \frac{\eta_S^\theta \eta_O^\theta}{\sqrt{1+x^2}} \begin{bmatrix} u_O & -1 \\ 1 & u_O \end{bmatrix} \begin{bmatrix} 1 & -x \\ -x & -1 \end{bmatrix} \begin{bmatrix} -u_S & 1 \\ 1 & u_S \end{bmatrix}, \tag{125}$$

where $u_S := \eta_S^\phi/\eta_S^\theta$, $u_O := \eta_O^\phi/\eta_O^\theta$.

The matrix $N_O R N_S^{-1}$ takes the form:

$$N_O R N_S^{-1} = \begin{bmatrix} \cos\mathcal{X} - \sin\mathcal{X} \\ \sin\mathcal{X} \quad \cos\mathcal{X} \end{bmatrix} \tag{126}$$

where

$$\sin\mathcal{X} := (1+X^2)^{-1/2}(1+x^2)^{-1/2}(X-x) \tag{127}$$

and the parameter X is defined as follows:

$$X := -\frac{u_S + u_O}{1 - u_S u_O}. \tag{128}$$

The quantities u_S and u_O are related to the angular positions of the observer and the source as follows:

$$u_S = \sin\theta_S[-\cot\theta_S\cos(\phi_O - \phi_S) + \cot\theta_O]/\sin(\phi_O - \phi_S), \tag{129}$$

$$u_O = \sin\theta_O[\cot\theta_O\cos(\phi_O - \phi_S) - \cot\theta_S]/\sin(\phi_O - \phi_S). \tag{130}$$

Let us a give a computation. For the exact solution of the lens equations we presented for an observer located at $\theta_O = \pi/3, \phi_O = 0$, with parameters for the Kerr black hole: $a = 0.9939, \mathcal{Q} = 25.64563, \Phi = -3.11$, we compute for the parameter $x = -0.0498019$ and $X = 0.0566708$.Consequently $\mathcal{X} = 0.106371$.

17 Conclusions.

- The precise analytic treatment of Kerr and Kerr-de Sitter black holes as gravitational lenses has been achieved. Full analytic strong-field calculation for the magnification factors was performed.

- Λ does contribute to the gravitational bending of light.

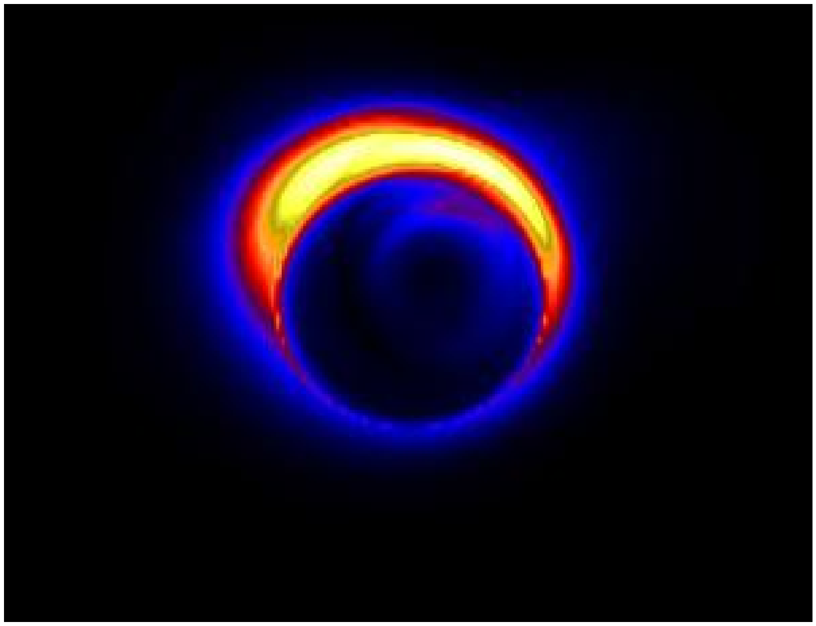
- Important application to the Sgr A* galactic centre black hole (Ghez). Closed form solutions for the periastron advance, frame-dragging and orbital periods for the observed orbits of S-stars (Kraniotis CQG 24 2007p1775) in conjuction with the dedicate measurements (Ghez *et al* 2008, Genzel *et al* 2010) already constrain significantly the mass of the galactic black hole, and they will eventually determine with high-precision the Sgr A* mass, as well as the distance to the galactic centre. Relativistic observables also depend sensitively on the spin of the black hole. Thus, gravitational lensing in conjuction with the observation of the relativistic effects from timelike geodesy provides a complementary and full test of General Relativity at the strong field regime.

- We reported new preliminary results concering the propagation of the polarization vector in Kerr spacetime. We made use of the Walker-Penrose constant in combination with our closed form solutions of the lens equations and computed novel strong field results about the change of the polarization vector and the rotation of the polarization plane.

- Fruitfull synergy of various fields of Science: general relativity, astrophysics, cosmology, pure mathematics(Pi1),(Pi2).

- If, the now under development, near-infrared intereferometric GRAVITY experiment (leader R. Genzel for the German team) and the proposed and approved thirty metre telescope (TMT) with leader A. Ghez for the US team, reach the aimed accurary of 10 μarcs and in combination with Very Long Baseline Interferometry (VLBI) observations, then it may be possible for these experiments to detect the effects of strong light bending (the shadow) by the galactic centre black hole. The observation of this shadow would be direct evidence of an event horizon. The departure from complete circularity it will tell us that Sagittarious A* is a spinning black hole.

END, Thank you for your time and attention.



The proof of the shadow of the black hole.