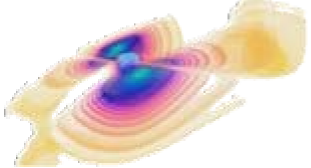

Iteration stability of the Einstein constraints in the conformal thin-sandwich approach

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Bernd Brügmann (FSU Jena)

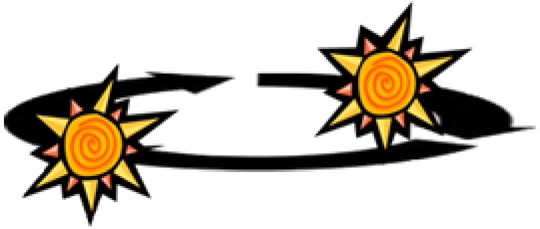
NEB 15
Chania
June 23, 2012

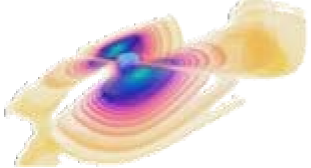
Iteration stability - Applications

- Single, non rotating star

- Single, rotating star

- Binary system





Articles where methods are used

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- U. M. Schaudt, Ann. Henri Poincaré 1 (2000) 945 – 976
- U. M. Schaudt & H. Pfister, Phys. Rev. Lett. 77, 16 (1996)
- …

Equations of Stellar Structure

Poisson

$$\Phi(\vec{r}) = -G \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

Euler

$$\nabla\left(h + \Phi - \tfrac{1}{2} v^2\right) = 0$$

EOS

$$h \coloneqq \int_0^{\rho} \frac{dP}{\rho}$$

Equations of Stellar Structure

Poisson
$$\Phi(\vec{r}) = -G \int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

Euler
$$h + \Phi - \frac{1}{2} v^2 = \kappa$$

EOS
$$\rho = \rho[h]$$

Iterative procedure (SCF method)

Poisson

$$\Phi(\vec{r}) = -G \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

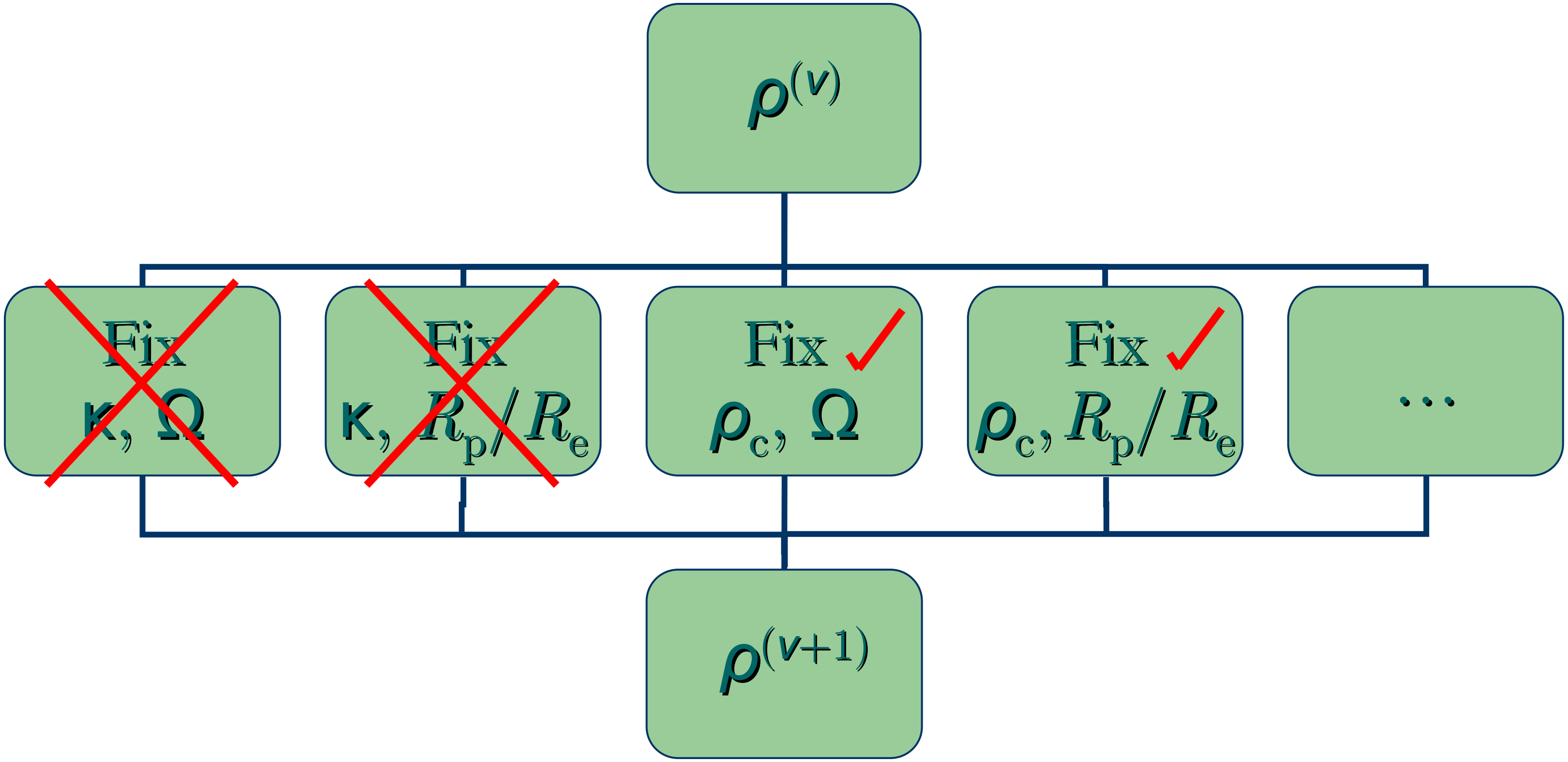
Euler

$$h + \Phi - \frac{1}{2} v^2 = \kappa$$

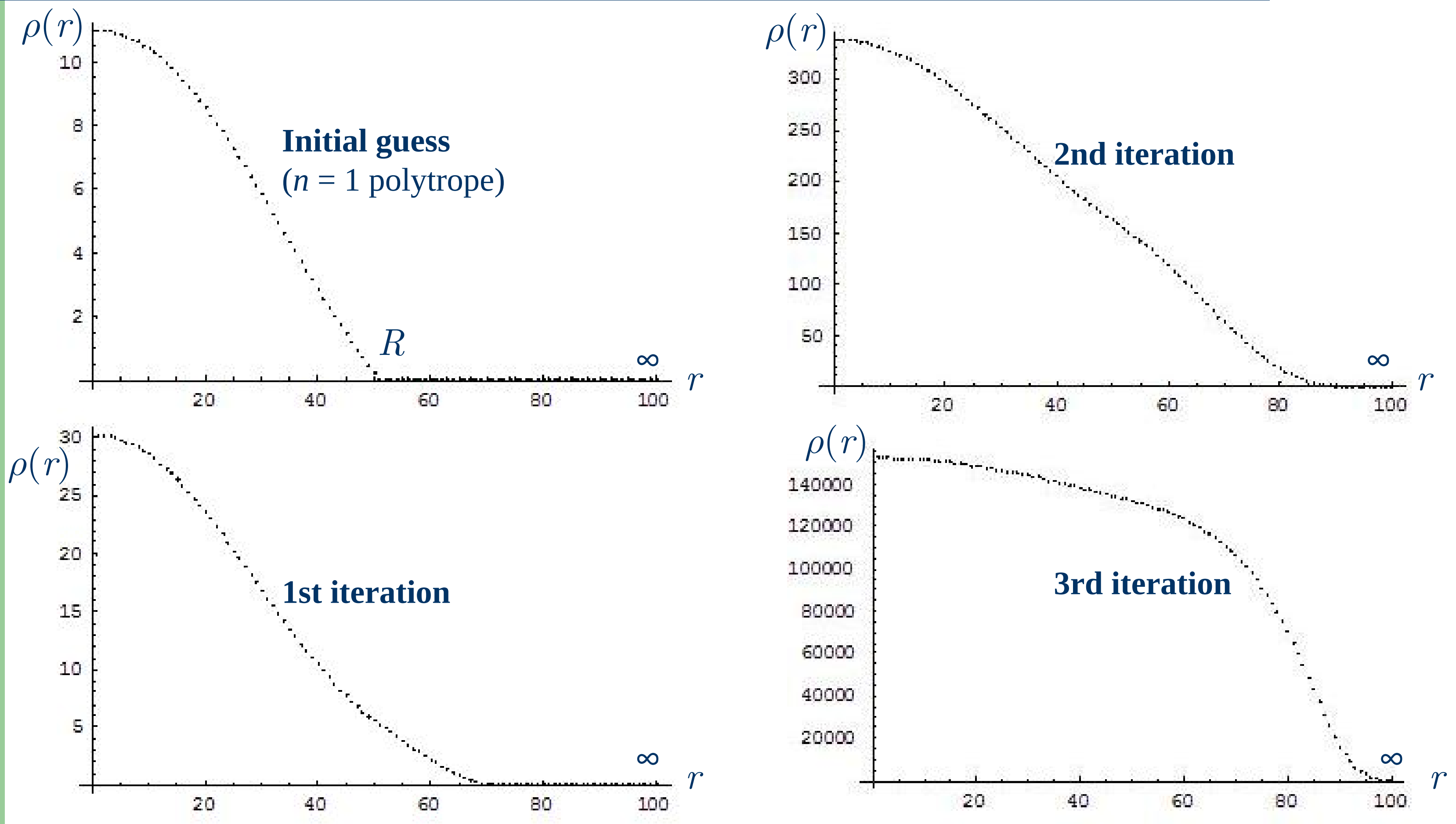
EOS

$$\rho = \rho[h]$$

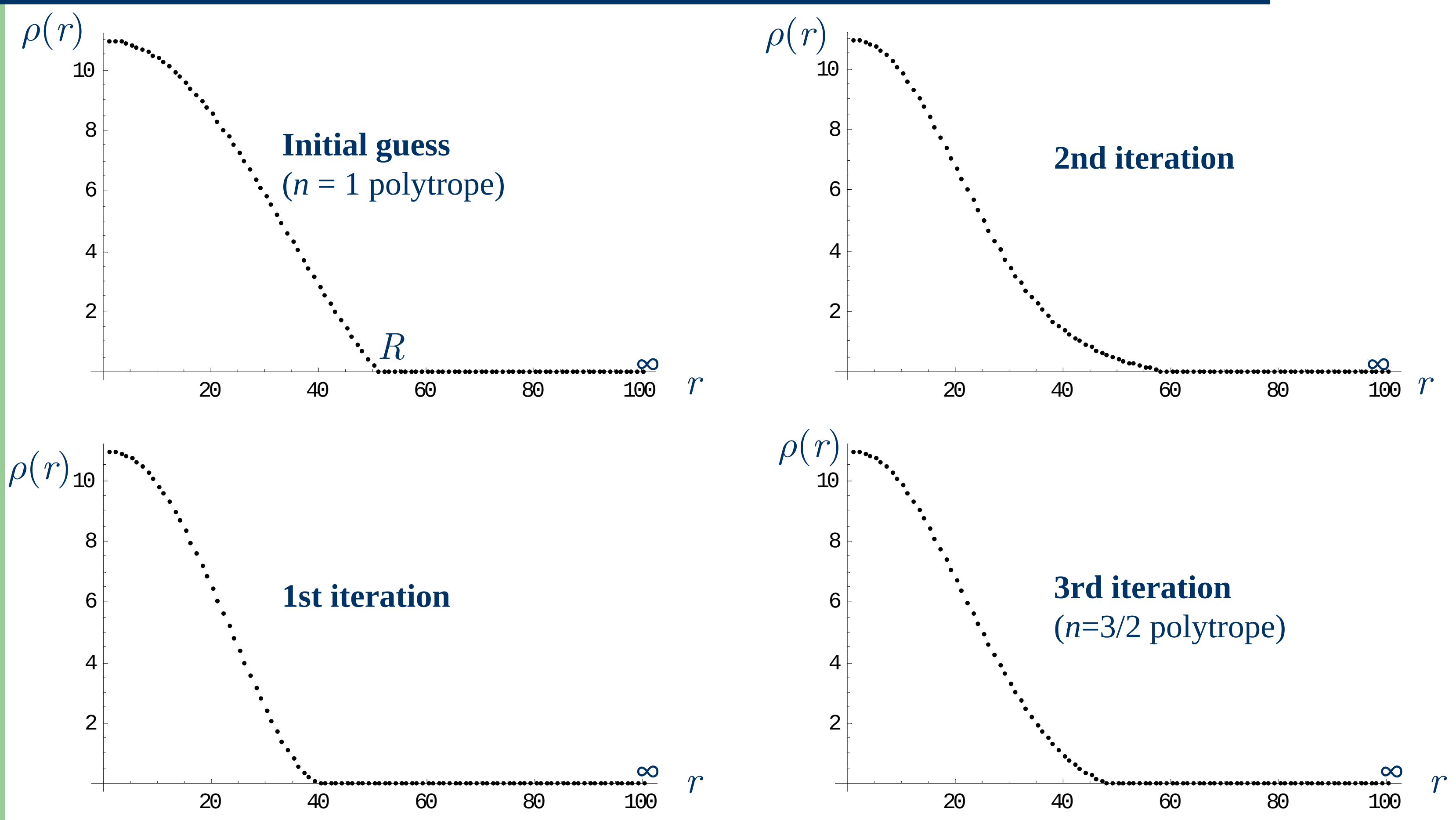
Variants of the SCF method



Iteration with fixed κ



Iteration with fixed central density



Iteration Stability

- SCF method has been used for > 40 years
- Little is known about why it works when one quantity is fixed and not another…
- In the past, obtaining numerical solutions with SCF was too easy for people to care why it worked…
- Not the case for relativistic binaries, so understanding convergence is crucial today…
- Consider behavior of iterative scheme near an exact solution

Iteration Stability

Exact solution

$$\Phi(\vec{r}) = -\tfrac{1}{4\pi}\int d^3r'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\eta + \Phi - \tfrac{1}{2}v^2 = \kappa$$

$$\rho = \eta^n$$

Linear perturbation

$$\delta\Phi(\vec{r}) = -\tfrac{1}{4\pi}\int d^3r'\,\frac{\delta\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\delta\eta + \delta\Phi = \delta\kappa$$

$$\delta\rho = n\eta^{n-1}\delta\eta$$

Iteration operator

$$\eta^{\text{new}}(\vec{r}) = \kappa + \tfrac{1}{4\pi}\int d^3r'\,\frac{[\eta^{\text{old}}(\vec{r}\,')]^n}{|\vec{r}-\vec{r}\,'|}$$

Iteration Stability

Exact solution

$$\Phi(\vec{r}) = -\tfrac{1}{4\pi}\int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\eta + \Phi - \tfrac{1}{2}v^2 = \kappa$$

$$\rho = \eta^n$$

Linear perturbation

$$\delta\Phi(\vec{r}) = -\tfrac{1}{4\pi}\int d^3r' \, \frac{\delta\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\delta\eta + \delta\Phi = \delta\kappa$$

$$\delta\rho = n\eta^{n-1}\delta\eta$$

Linear iteration operator

$$\delta\eta^{\rm new}(\vec{r}) = \delta\kappa + \frac{n}{4\pi}\int d^3r' \eta^{n-1}(\vec{r}\,')\,\frac{\delta\eta^{\rm old}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Iteration Stability

Linear iteration operator

$$\delta\eta^{\mathrm{new}}(\vec{r}) = \delta\kappa + \frac{n}{4\pi}\int d^3r'\,\eta^{n-1}(\vec{r}')\,\frac{\delta\eta^{\mathrm{old}}(\vec{r}')}{|\vec{r}-\vec{r}'|} \equiv L\left(\delta\eta^{\mathrm{old}}\right)$$

Fixed-κ iteration: $\delta\kappa = 0$

Fixed-$\eta(0)$ iteration: $\delta\eta(0) = 0$

Iteration Stability

Linear iteration operator

$$\delta\eta^{\text{new}}(\vec{r}) = \delta\kappa + \frac{n}{4\pi}\int d^3r'[\eta(\vec{r}')]^{n-1}\frac{\delta\eta^{\text{old}}(\vec{r}')}{|\vec{r}-\vec{r}'|} \equiv L\left(\delta\eta^{\text{old}}\right)$$

Eigenvalue problem

$$L\left(\delta\eta\right) = \lambda\delta\eta$$

Necessary condition for convergence: $|\lambda| < 1$

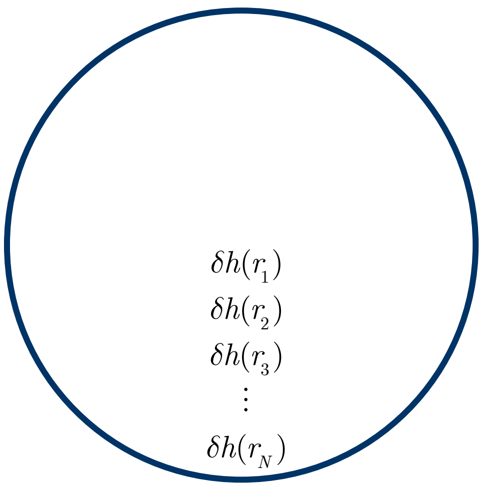
Iteration with fixed κ

Linear iteration operator

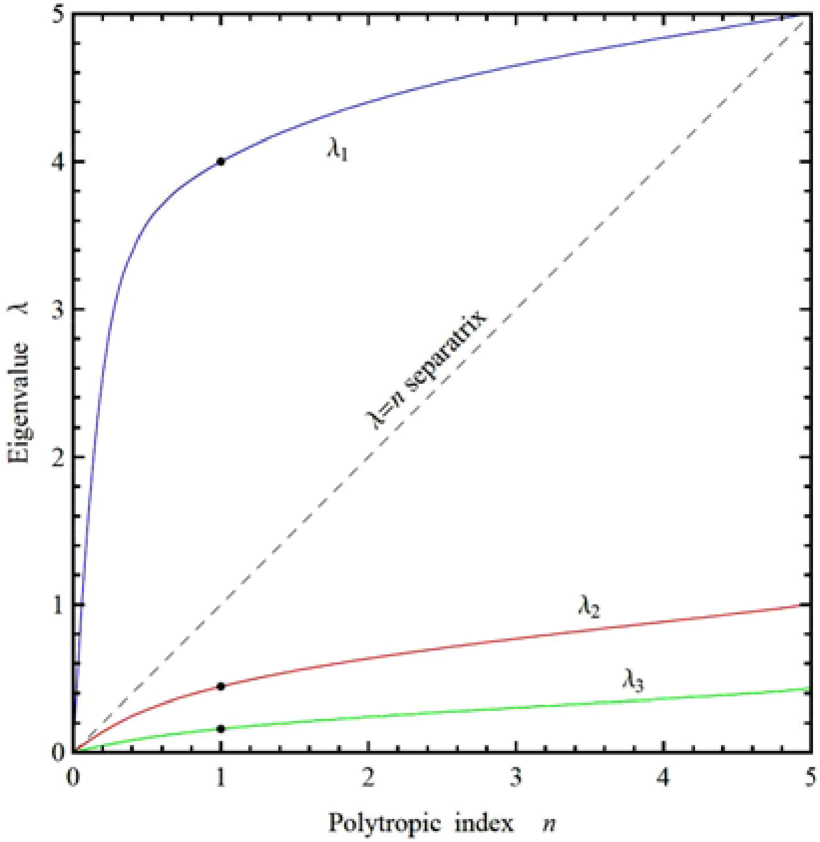
$$\delta h^{\text{new}} = G \int d^3r' \, \frac{d\rho}{dh} \frac{\delta h^{\text{old}}}{|\vec{r} - \vec{r}'|} \equiv L\left(\delta h^{\text{old}}\right)$$

Discretization

$$\delta h_i^{\text{new}} = \sum_j L_{ij} \delta h_j^{\text{old}}$$



Iteration with fixed κ



All eigenvalues < 1, except $\lambda_1 \geq n$

Iteration with fixed central density

$$\delta h^{\text{new}} = G \int d^3 r' \, \frac{d\rho}{dh} \left(\frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r}'|} \right) \delta h^{\text{old}} \equiv L\left(\delta h^{\text{old}}\right)$$

Discretization

$$\delta h_i^{\text{new}} = \sum_j L_{ij} \delta h_j^{\text{old}}$$

Iteration with fixed central density

Jordan form of **L**

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

Iteration with fixed central density

Jordan form of L

$$\mathbf{J}^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & \ddots & \\ & & 0 & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$$

Iteration with fixed central density

Jordan form of **L**

$$\mathbf{J}^{N-1} = \begin{pmatrix} 0 & & \cdots & 0 & 1 \\ & 0 & & & 0 \\ & & 0 & & \vdots \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

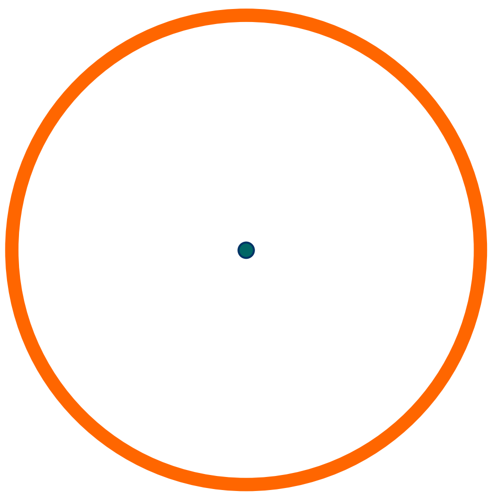
Iteration with fixed central density

Jordan form of L

$$\mathbf{J}^N = \begin{pmatrix} 0 & & \cdots & 0 & 0 \\ & 0 & & & 0 \\ & & 0 & & \vdots \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

Explanation

Fixed-ρ_o linear iteration - explanation



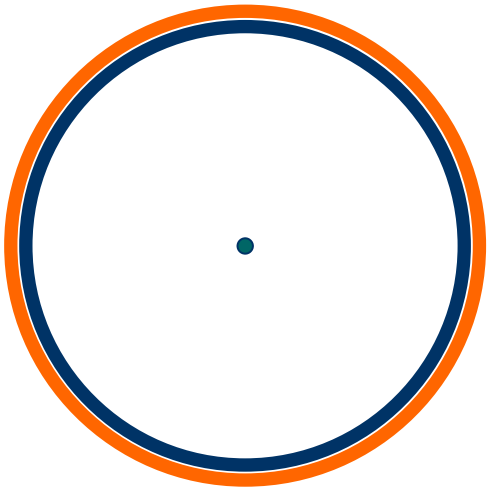
$$\delta\Phi(\vec{r}) = -\frac{1}{4\pi}\int d^3r' \frac{\delta\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\delta\eta + \delta\Phi = \delta\kappa$$

$$\delta\rho = \frac{d\rho}{d\eta}\delta\eta$$

Explanation

Fixed-ρ_o linear iteration - explanation



$$\delta\Phi(\vec{r}) = -\frac{1}{4\pi}\int d^3r' \, \frac{\delta\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

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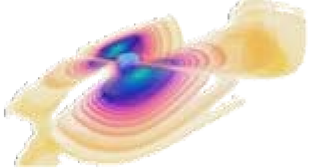


Einstein constraints

Accurate calculation of physically relevant initial data corresponding to a system of black holes and/or neutron stars is an essential part of binary inspiral simulations.

In the 3+1 formalism, Einstein's equations are solved as a Cauchy-problem with initial data prescribed on a spacelike hypersurface. These initial data must satisfy the constraint equations. However, iterative algorithms for solving these equations may diverge, or may converge to an unphysical solution…

Euler – XCTS system

Hamiltonian constraint $\nabla^2\Psi = -\frac{1}{2}\epsilon\Psi^5$

Maximal slicing condition $\nabla^2(\alpha\Psi) = \frac{1}{2}(\epsilon + 6p)\alpha\Psi^5$

1st integral to Euler equation $\alpha h = \kappa$

EOS $\epsilon = \epsilon[h], \quad p = p[h]$

Euler – XCTS system

Hamiltonian constraint

$$\Psi(\vec{r}) = 1 - \frac{1}{8\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\epsilon(\vec{r}')\Psi^5(\vec{r}')$$

Maximal slicing condition

$$\alpha(\vec{r})\Psi(\vec{r}) = 1 + \frac{1}{8\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}[\epsilon(\vec{r}') + 6p(\vec{r}')]\alpha(\vec{r}')\Psi^5(\vec{r}')$$

1st integral to Euler equation

$$\alpha h = \kappa$$

EOS

$$\epsilon = \epsilon[h], \quad p = p[h]$$

Euler – XCTS system

Hamiltonian constraint

$$\Psi(\vec{r}) = 1 - \frac{1}{8\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\epsilon(\vec{r}')\Psi^5(\vec{r}')$$

Similar to updating of Newtonian enthalpy for fixed κ

$$\eta(\vec{r}) = \kappa + \frac{1}{4\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\eta^n(\vec{r}')$$

Newtonian experience suggests eigenvalues > 1,
but $\epsilon(\vec{r})$ has compact support

Euler – XCTS system

Hamiltonian constraint

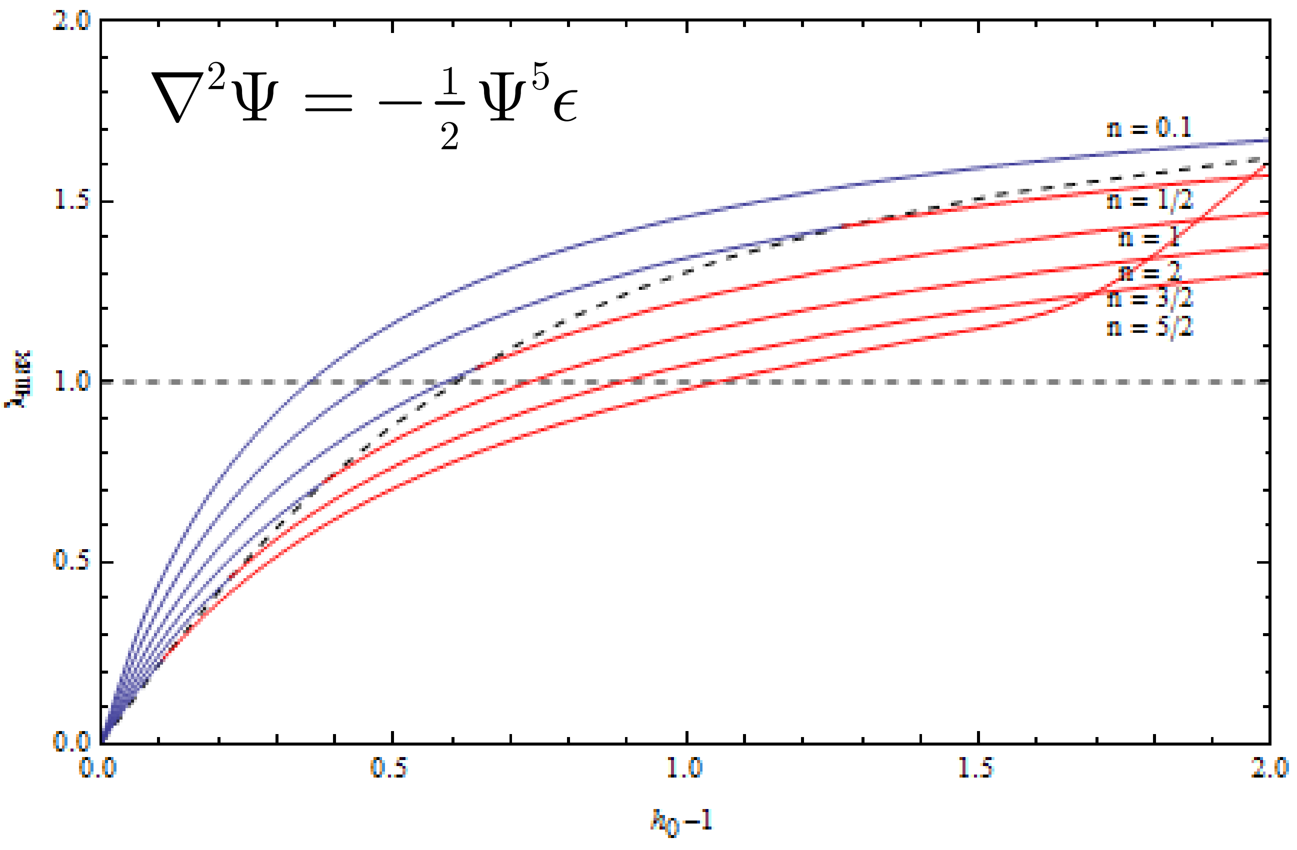
$$\delta\Psi(\vec{r}) = -\tfrac{5}{8\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\epsilon(\vec{r}')\Psi^4(\vec{r}')\delta\Psi(\vec{r}')$$

Similar to updating of Newtonian enthalpy for fixed κ

$$\delta\eta(\vec{r}) = \frac{n}{4\pi}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\eta^{n-1}(\vec{r}')\delta\eta(\vec{r}')$$

Newtonian experience suggests eigenvalues > 1,
but $\epsilon(\vec{r})$ has compact support

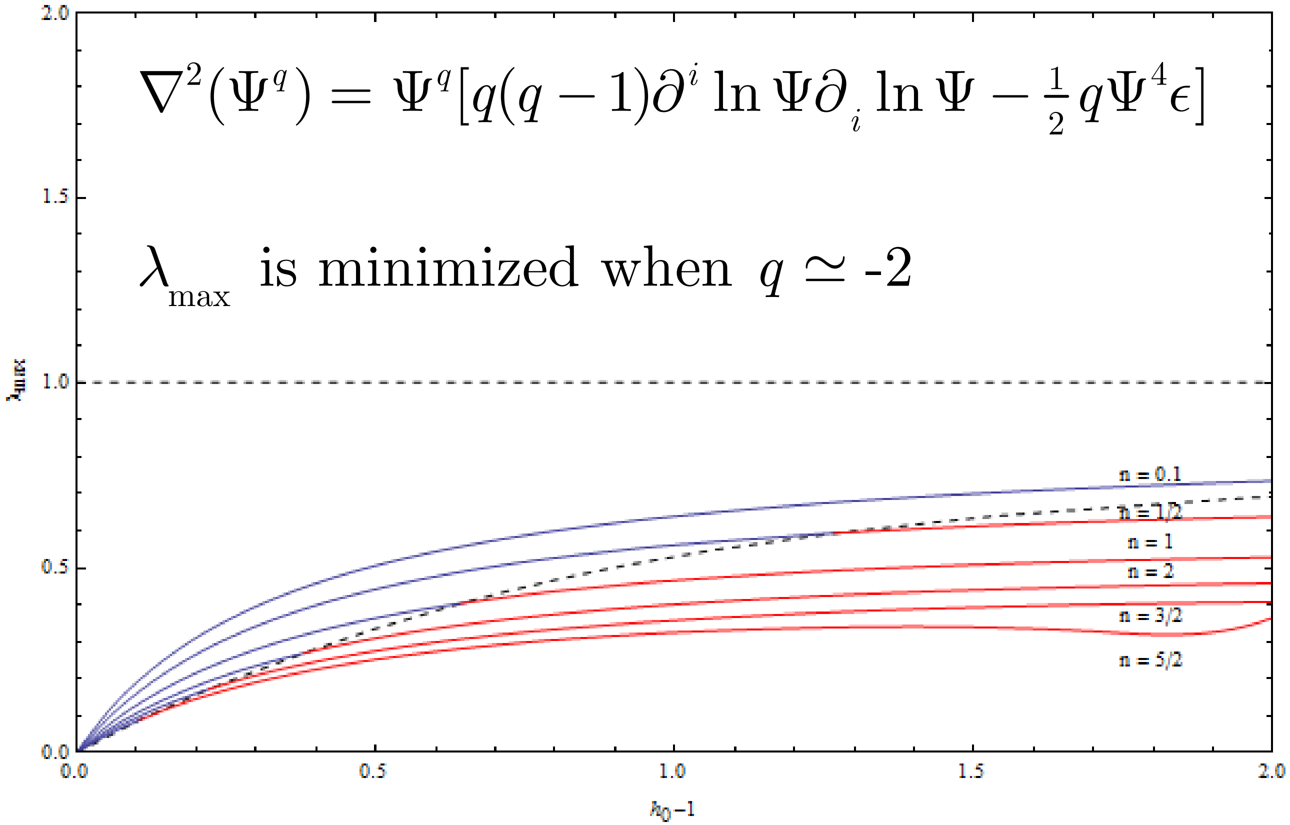
Hamiltonian constraint: iteration stability



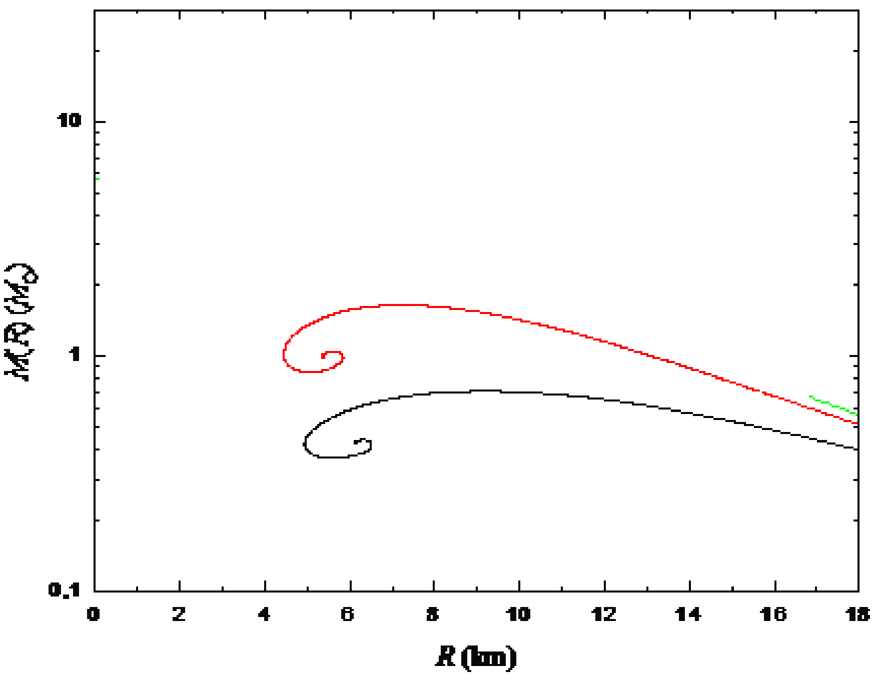
Hamiltonian constraint: iteration stability

$$\nabla^2(\Psi^q) = \Psi^q[q(q-1)\partial^i \ln\Psi\partial_i \ln\Psi - \tfrac{1}{2}q\Psi^4\epsilon]$$

λ_{max} is minimized when $q \simeq$ -2



Turning point stability



Iteration Stability

Newtonian problem (completed):

R.H. Price, C. Markakis, J.L. Friedman

Iteration stability for simple Newtonian stellar systems

J. Math. Phys. 50, 073505 (2009)

Relativistic problem (in progress):

C. Markakis et al

Einstein constraints & fluid equilibria: optimizing convergence & stability of the extended conformal thin-sandwich system

- Study effect of rescaling source terms on stability
- Analyze full Euler-XCTS system
- Convergence optimization