Using Noether symmetries to specify f(R) gravity

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The Simplicity of Lambda

In a FRW flat spacetime the field equations of the ΛCDM

$$3\left(\frac{\dot{a}}{a}\right)^2+\Lambda = \frac{\rho_{0m}}{a^3} \quad (1)$$

$$\ddot{a}+\frac{1}{2a}\dot{a}^2-\frac{1}{2}\Lambda a = 0 \quad (2)$$

They describe the Hamiltonian system of the 1-D hyperbolic oscillator. The Solution of the field equations is

$$a(t)=a_0\sinh^{\frac{3}{2}}\omega t$$

That means that eq. (2) admits eight Lie point symmetries which is the $sl(3,R)$ algebra. Therefore the system of eqs. (1),(2) admits five Noether point symmetries, as many as the free particle. The field equations for the: a) empty space b) CDM c) de Sitter and d) ΛCDM admit the same algebra of Lie/Noether symmetries. But not the same representation!

f(R) gravity

The action for the $f\,(R)$ gravity is

$$S = \int dx^4 \sqrt{-g} f\,(R) + \int dx^4 \sqrt{-g} L_{matter} + \text{Boundary Terms}$$

where L_{matter} is the Lagrangian of matter. The modified field equations are

$$G_{(Mod)\ \nu}^{\ \mu} = (1 + f^{'}) G_{\nu}^{\mu} - g^{\mu\alpha} f_{R,\alpha\,;\nu} + \left[\Box f^{'} - \frac{1}{2}(f - R f^{'}) \right] \delta_{\nu}^{\mu} = k^2\, T_{\nu}^{\mu}$$

In the content of a FRW C (with zero spatial curvature)

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}dx^i dx^k$$

with a dust fluid $(p_m = 0)$ and for comoving observers $u^a = \partial_t$, $u^a u_a = -1$ the field equations become

$$\begin{aligned}
3f^{'}H^2 &= k^2\rho_m + \frac{f^{'}R - f}{2} - 3Hf^{''}\dot{R} \\
2f^{'}\dot{H} + 3f^{'}H^2 &= -2Hf^{''}\dot{R} - \left(f^{'''}\dot{R}^2 + f^{''}\ddot{R}\right) - \frac{f - Rf^{'}}{2} \\
R &= 6(2H^2 + \dot{H})
\end{aligned}$$

From $T^{\mu\nu}_{;\nu} = 0$ we find $\rho_m = \rho_{m0}a^{-3}$.

In order an $f(R)$ model to be cosmologically viable must satisfy the following conditions (Amendola L and Tsujikawa S Dark Energy Theory and Observations)

- $f'(R) > 0$ for $R \succeq R_0 > 0$ where R_0 is the Ricci scalar at the present epoch. If the final attractor is a de Sitter point there needs to be $f'(R) > 0$ for $R \succeq R_1 > 0$ where R_1 is the Ricci scalar at the de Sitter points.
- $f''(R) > 0$ for $R \succeq R_0$, $f(R) \rightarrow R - 2\Lambda$ for $R \gg R_0$ in order to be consistent with local gravity tests and for the presence of the matter dominated era.
- $0 < \frac{Rf''}{f'}(r = -2) < 1$ where $r = -\frac{Rf'}{f} = -2$ for the stability of the late de Sitter point.

Some $f(R)$ models are

- The Starobinsky model
  $f(R) = R - mR_c\left[1 - \left(1 + R^2/R_c^2\right)^{-n}\right]$ (Starobinsky A A 2007 JETP 86 157)
- The Tsujikawa model
  $f(R) = R - mR_c \tanh\left(R/R_c\right)$ (Tsujikawa S 2008 Phys. Rev. D 77 023507)
- The generalization of the ΛCDM $f(R) = \left(R^b - 2\Lambda\right)^c$, $c \succeq 1$. (Amendola L et.al 2008 Phys. Lett. B 660 125)
- and the list goes on...

- The modified field equations describe a 2-D Hamiltonian dynamical system of second order ODE.
- It is proposed that the field equations should be integrable via point symmetries.
- In order for this to be achieved, $f(R)$ will be defined so that the dynamical system admits Noether point symmetries. This is a geometric criterium since the point symmetries are generated from the mini superspace of the field equations. (Tsamparlis & Paliathanasis arXiv:1101.5771)

The Dynamical system

The Lagrangian of the field equations is

$$L\left(a,\dot{a},R,\dot{R}\right)=6af^{'}\ \dot{a}^{2}+6a^{2}f^{''}\ \dot{a}\dot{R}+a^{3}\left(f^{'}R-f\right) \tag{3}$$

The Lagrangian is of the form

$$L=T-V$$

in the space of the variables $\{a,R\}$. T is the Kinetic term

$$T=6af^{'}\ \dot{a}^{2}+6a^{2}f^{''}\ \dot{a}\dot{R}$$

and V the potential

$$V=-a^{3}\left(f^{'}R-f\right)$$

The Dynamical system

The Lagrangian (3) is autonomous and admits as Noether point symmetry ∂_t with Noether Integral the Hamiltonian

$$E = 6af^{'}\ \dot{a}^2 + 6a^2 f^{''}\ \dot{a}\dot{R} - a^3\left(f^{'}R - f\right)$$

which is the modified Friedmann equation. The constant E is related to the density of the dust fluid as follows

$$E = 6\Omega_m H_0^2.$$

Noether Symmetries

If $X = \xi\left(t, x^k\right)\partial_t + \eta^i\left(t, x^k\right)\partial_i$ is the generator of a Lie symmetry then X is a Noether symmetry if the following condition holds

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt}$$

The solution of the Noether Condition is (see Tsamparlis & Paliathanasis arXiv:1101.5771) the Homothetic algebra of the kinetic 2-D C

$$ds^2_{(2)} = 12af'da^2 + 12a^2 f'' dadR$$

The Ricci scalar $R_{(2)} = 0$ and since all 2-D spaces are Einstein spaces, hence $ds_{(2)}$ is a flat space.
This means that the Homothetic algebra is the one of a flat space which consist of 2 gradient KVs, a non-gradient KV and a gradient HV.

Noether Symmetries

For the modified field equations to admit extra Noether symmetries other than the trivial ∂_t we found that there are two categories of $f(R)$.

- The Power Law models (Capozziello et.al Phys. Lett. B 639)
  The $f(R) = R^{\frac{3}{2}}$ admits three extra Noether symmetries.
  The $f(R) = R^{\frac{7}{8}}$ admits two extra Noether symmetries, $sl(2,R)$.
  The $f(R) = R^{n}$ $\left(n \neq 1, \frac{3}{2}, \frac{7}{8}\right)$ admits one extra Noether symmetry.
- The Λ_{bc}CDM models
  The $f(R) = (R-2\Lambda)^{\frac{3}{2}}$ $b=1, c=3/2$ admits two extra Noether symmetries and the equivalent Newtonian dynamical system is the anisotropic forced oscillator.
  The $f(R) = (R-2\Lambda)^{\frac{7}{8}}$ $b=1, c=7/8$ admits two extra Noether symmetries $sl(2,R)$ and the equivalent Newtonian dynamical system is the Ermakov-Pinney system.

Analytic Solution for b=1 , c= 3/2

In that case the Lagrangian of the modified field equations is

$$L = 9a\left(R-2\Lambda\right)^{\frac{1}{2}}\dot{a}^2 + \frac{9a^2}{2}\left(R-2\Lambda\right)^{-\frac{1}{2}}\dot{a}\dot{R} + \frac{a^3}{2}\left(R+4\Lambda\right)\left(R-2\Lambda\right)^{\frac{1}{2}}$$

Changing now the variables from (a, R) to the normal coordinates (x, y) via the relations $a = (9/2)^{-\frac{1}{3}}\sqrt{x}$, $R = 2\Lambda + y^2/x$ the Lagrangian becomes

$$L = \dot{x}\dot{y} + V_0\left(y^3 + \bar{m}xy\right)$$

and the extra Noether First Integrals are

$$I_{\pm} = e^{\pm\omega t}\dot{y} \mp \omega e^{\pm\omega t}y$$

From these the following time independent first integral is constructed

$$\Phi = I_+ I_- = \dot{y}^2 - \omega^2 y^2$$

Analytic Solution for b=1 , c= 3/2

The solution is

$$x(t) = x_{1G} e^{\omega t} + x_{2G} e^{-\omega t} + \frac{1}{4\bar{m}\omega^2} \left(l_2 e^{\omega t} + l_1 e^{-\omega t} \right)^2 + \frac{l_1 l_2}{\bar{m}\omega^2}.$$

$$y(t) = \frac{l_2}{2\omega} e^{\omega t} - \frac{l_1}{2\omega} e^{-\omega t}$$

and the Hamiltonian Constrain gives $E = \omega \left(x_{1G} l_1 - x_{2G} l_2 \right)$.
By inserting the analytical solution into the modified Friedmann equation, it can be easily demonstrated that in the matter dominated era the Hubble parameter tends to its nominal form, namely

$$H(a) \rightarrow a^{-3/2}$$

Analytic Solution for b=1 c=7/8

In this case, the normal coordinates are (u, v) where $a = \sqrt{uv}, R = 2\Lambda + \frac{v^{12}}{u^4}$. Under a coordinate transformation the Lagrangian becomes

$$L = \dot{u}^2 - u^2 v^{-2} \dot{v}^2 + \lambda u^2/4 + 4V_0 v^{12} u^{-2}$$

This is the Ermakov-Pinney system and the general solution is

$$\begin{aligned} u(t) &= \left(u_1 e^{2\lambda t} + u_2 e^{-2\lambda t} + 2u_3\right)^{\frac{1}{2}} \\ v(t) &= 2^{\frac{1}{6}} \phi^{\frac{1}{12}} e^{-A(t)} \left(4V_0 + e^{-12A(t)}\right)^{-\frac{1}{6}} \end{aligned}$$

where $A(t) = \arctan\left(\frac{2\lambda}{\sqrt{\phi}}\left(u_1 e^{2\lambda t} + u_3\right)\right) + 4\lambda^2 u_1 \sqrt{\phi}$, $E = -2\lambda u_3$ and ϕ is the Lewis Invariant.

FRW with non zero Spatial Curvature

In the case of the non zero spatial curvature FRW spacetime the $f(R)$ models which admit extra Noether symmetries are

- $f(R) = R^2$
- $f(R) = R^{\frac{3}{2}}$
- $f(R) = (R - 2\Lambda)^{\frac{3}{2}}$
  The analytical solution for the $\Lambda_{1,\frac{3}{2}}$CDM model is

$$a^2(t) = x(t) = x_{flat}(t) + \frac{\bar{K}}{\omega^2}$$

where $x_{flat}(t)$ is the analytical solution for the flat case for the same $f(R)$ function.

The Λ_{bc}CDM model was phenomenologically selected in order to extend the concordance Λ cosmology.
It appears from the current analysis that it has a geometrical basis.
For $b = 1, c = 3/2$ it provides a cosmic history which is similar to those of the usual dark energy models while at the same time there provides an analytical solution for all spatial curvature models.