ROTATING FIGURES OF EQUILIBRIUM

NEB XV

In the context of Newtonian theory of Gravitation

- The study of the shape of a rotating homogeneous mass ,with uniform velocity has been carried out by some of the greatest scientists :
- I) Newton (Principia, Book III,Proposition XVIII-XX), small eccentricity.
- II) Maclaurin ,the eccentricity is not small

  Maclaurin Spheroids (1742)

- III) Jacobi, (1834) , triaxial ellipsoids
- IV) Meyer and Liouville (1842,1846) relation between Maclaurin spheroids and Jacobi ellipsoids ,no figures of equilibrium are possible when the angular velocity exceeds a limit.
- V) Dirichlet, Dedekind, Riemann (1892)

Dirichlet (in a Lagrangian framework) and Dedekind working on the paper of Dirichlet defined the ellipsoids of Dirichlet. Riemann gave a complete solution for the stationary figures of equilibrium.

VI) Poincare(1885), Cartan (1924)

Stability of ellipsoids figures of equilibrium, (pear shaped configurations), stability of Jacobi ellipsoids

Details and the continuation of the story in :

"Ellipsoidal Figures of Equilibrium" by S Chandrasekhar

In the context of General Relativity

- Problem

  Find all possible equilibrium interior configurations, satisfying Einstein's equations, bounded by a surface of zero hydrostatic pressure, matched across this surface to a vacuum solution ,sharing the same symmetries: stationarity and axial symmetry.

- We suppose that the space admits an isometry Abelian group invertible, with non null surface of transitivity

$$ds^2 = g_{tt}dt^2 + 2g_{tz}dtdz + g_{zz}dz^2 - g_{xx}dx^2 - g_{yy}dy^2$$

- The components of the metric tensor depend only on x and y, the group is generated by $\frac{\partial}{\partial t}$ (time-like Killing vector, implying the stationarity) and $\frac{\partial}{\partial z}$ (space-like Killing vector, implying the axial symmetry)

Symmetric null tetrad and Carter's metric [A]

$$ds^2 = (Ldt + Mdz)^2 - (Ndt + Pdz)^2 - S^2dx^2 - R^2dy^2$$

, L, M, N, P are functions of x and y

$$ds^2 = (x^2 + y^2)\left\{\frac{E^2(y)}{(x^2 + y^2)^2}(dt - x^2dz)^2 - \frac{H^2(x)}{(x^2 + y^2)^2}(dt + y^2dz)^2 - \frac{x^2dx^2}{F^2(x)} - \frac{y^2dy^2}{G^2(y)}\right\}$$

Kerr and Whalquist metrics

$G^2(y) = y^2 E^2(y)$ $\qquad$ $F^2(x) = x^2 H^2(x)$

$E^2(y) = \frac{1}{2} b y^2 + d y + p$ $\qquad$ $H^2(x) = -\frac{1}{2} b x^2 + c x + p$

$b = 2$ $\qquad$ $d = -2m$ $\qquad$ $c = 0$ $\qquad$ $p = a^2$

$y = r$ $\qquad$ $x = a \cos\theta$

Whalquist metric

e+3p=constant

$$W(y) = \frac{G^2(y)}{E^2(y)} = k_4 y^4 + k_2 y^2 - k_0$$

$$Z(x) = \frac{F^2(x)}{H^2(x)} = -k_4 x^4 + k_2 x^2 + k_0$$

Surfaces of revolution

- We consider three dimensional Euclidean space foliated with surfaces of revolution with centre O and common axis of rotation L. Each surface is obtained by revolving a plane curve C about the axis of rotation L, this axis coincides with the axis of rotation of the fluid configuration. In a Cartesian coordinate system ,the curve C lies in the plane and is the axis of revolution. In this coordinate system, a parametric representation of the curve C is defined as follows:

Parametric representation of a surface of revolution

$$x_1 = h_1(t)\cos\Phi \qquad x_2 = h_1(t)\sin\Phi \qquad x_3 = h_2(t)$$

$$x_1 = h_1(r,\theta)\cos(\Phi), x_2 = h_2(r,\theta)\sin(\Phi), x_3 = r\cos(\theta)$$

$$ds_3^2 = (h_{1r}^2 + \cos(\theta)^2)dr^2 + (h_{1\theta}^2 + r^2\sin(\theta)^2)d\theta^2 + h_1^2 d\Phi^2 +$$

$$+2(h_{1r}h_{1\theta} - r\sin(\theta)\cos(\theta))drd\theta$$

$$h_{1\theta} = \frac{\vartheta h_1}{\vartheta\theta}$$

Euclidean and Riemannian metrics

$$ds_3^2 = \frac{[a^2 h_{1\,r}^2 + \cos(\theta)^2]}{f^2(r,\theta)} dr^2 + [h_{1\,\theta}^2 + r^2 \sin(\theta)^2] d\theta^2 + h_1^2 d\Phi^2$$

$$f(r,\theta) = 1 \qquad \textit{Euclidean}$$

$$y = r \qquad x = a\cos(\theta)$$

$$ds_3^2 = \frac{[a^2 h_{1y}^2 + x^2]}{a^2 f^2(x,y)} dy^2 + [a^2 h_{1x}^2 + y^2] dx^2 + h_1^2 d\Phi^2$$

Quotient space of the comoving observers

$$u = u^i \frac{\vartheta}{\vartheta x^i} = U(x,y)\frac{\vartheta}{\vartheta t} + V(x,y)\frac{\vartheta}{\vartheta z} \qquad u_i u^i = 1$$

$$ds_3^2 = \frac{[a^2 h_{1y}^2 + x^2]}{a^2 f^2(x,y)} dy^2 + [a^2 h_{1x}^2 + y^2] dx^2 + h_1^2 d\Phi^2 = g_{ij} dx^i dx^j - u_i u_j dx^i dx^j$$

$$\frac{[a^2 h_{1y}^2 + x^2]}{a^2 f^2} = g_{yy}, \frac{[a^2 h_{1x}^2 + y^2]}{a^2} = g_{xx}, V^2 [g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}] = h_1^2 \qquad dt - \frac{U}{V} d\varphi = d\Phi$$

$$\varphi = az$$

Carter's family [A]

$$ds^2 = (x^2 + y^2)\left\{\frac{E^2(y)}{(x^2+y^2)^2}(dt - x^2 dz)^2 - \frac{H^2(x)}{(x^2+y^2)^2}(dt + y^2 dz)^2 - \frac{x^2 dx^2}{F^2(x)} - \frac{y^2 dy^2}{G^2(y)}\right\}$$

$$\frac{[a^2 h_{1y}^2 + x^2]}{a^2 f^2} = (x^2 + y^2)\frac{y^2}{G^2}$$

$$\frac{[a^2 h_{1x}^2 + y^2]}{a^2} = (x^2 + y^2)\frac{x^2}{F^2}$$

$$V^2 E^2 H^2 = a^2 h_1^2$$

$$a^2 h_{1y} h_{1x} + xy = 0$$

The surface of revolution is completely defined

$$V = e_1 \frac{h_1 a}{EH} \qquad e_1 = \pm 1 \qquad e_2 = \pm 1$$

$$U = \frac{1}{E^2 - H^2}\{e_1 \frac{(x^2E^2 + y^2H^2)}{EH} h_1 + e_2[(x^2 + y^2)(E^2 - H^2) + h_1^2(x^2 + y^2)^{\frac{1}{2}}]\}$$

$$h_1 = \frac{1}{a}(a^2 - x^2)^{\frac{1}{2}}(a^2 + y^2)^{\frac{1}{2}} + g$$

$$F^2 = x^2(a^2 - x^2) \qquad \frac{G^2}{f^2} = y^2(a^2 + y^2)$$

Two possible cases

(I)

$g = 0$ $\qquad \dfrac{x_1^2 + x_2^2}{a^2 + r^2} + \dfrac{x_3^2}{r^2} = 1$ $\qquad$ *Krasinski* 1978

(II)

$g \neq 0$ $\qquad$ *torii of revolution* $\qquad \dfrac{(x_1 - g)^2}{a^2 + r^2} + \dfrac{x_3^2}{r^2} = 1$

Perfect fluid with heat flux

$$T_{ij} = (e+p)u_i u_j - p g_{ij} + q_i u_j + q_j u_i$$

$$2\left[x^3 y(x^2+y^2)^2\right]WE^2_{yy} + (x^2+y^2)\left[-2x^3(5y^2+x^2)W + x^3 y(x^2+y^2)W_y\right]E^2_y +$$

$$+8x^3y^3(W+Z)E^2 -$$

$$-2\left[xy^3(x^2+y^2)^2\right]ZH^2_{xx} - (x^2+y^2)\left[-2y^3(5x^2+y^2)Z + xy^3(x^2+y^2)Z_x\right]H^2_x -$$

$$-8x^3y^3(W+Z)H^2 = 0 \qquad W = \frac{G(y)}{E(y)} \qquad Z = \frac{F(x)}{H(x)}$$

Solution

$$W = kE^2 + k_4 y^4 + k_2 y^2 + k_0$$

$$Z = kH^2 + l_4 x^4 + l_2 x^2 + l_0$$

$$H^2 = \frac{a^2(a^2 - x^2)}{k_0 - a^2 k_2}$$