Revising the Multipole Moments of numerical spacetimes.

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• A few words on multipole moments.

• Numerical spacetimes for rapidly rotating neutron stars.

• Multipole moments of numerical spacetimes.

• Consequences of correcting the multipole moments.
A few words on multipole moments.
Why are we interested in multipole moments?

- Global properties of the spacetime (the multipole structure determines the geometry),

- Relevant in gravitational wave astronomy (the observables can be related to the moments of the source),

- Constraining the equation of state for NS (from the relation between $M, J, Q, \ldots$),

- Can be used to construct analytic spacetimes for the exterior of neutron stars.
Newtonian multipole moments:

\[
\Phi(r) = \frac{Q}{r} + \frac{Q_a x^a}{r^3} + \frac{Q_{ab} x^a x^b}{r^5} + \ldots
\]  

(1)

where, \( Q \), \( Q_a \), \( Q_{ab} \), are some integrals on the source

\[
Q = \int \rho d^3x, \quad Q_a = \int x^a \rho d^3x, \ldots
\]  

(2)

The multipole moments are generally tensorial quantities.
Definition of the moments at infinity:

\[ x^a \to \tilde{x}^a = r^{-2} x^a : \tilde{r}^2 = \tilde{x}^a \tilde{x}_a = r^{-2} \]

\[ \Phi(r) = \tilde{r} \left( Q + Q_a \tilde{x}^a + Q_{ab} \tilde{x}^a \tilde{x}^b + ... \right) \tag{3} \]

If we define the potential at infinity \( \bar{\Phi} = \tilde{r}^{-1} \Phi \) then the moments are

\[ P_{a_1...a_n} = \tilde{D}_{a_n} P_{a_1...a_{n-1}} = \tilde{D}_{a_1}...\tilde{D}_{a_n} \bar{\Phi} \tag{4} \]
Relativistic multipole moments:

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential" by a recursive relation,
- There are two sets of moments, the Mass moments and the Rotation moments,
- For the two sets of moments we have two generating potentials, $\Phi_M$, $\Phi_J$,

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.
Numerical spacetimes for rapidly rotating neutron stars.
In 1976 Butterworth and Ipser wrote the line element for a stationary and axially symmetric spacetime\(^1\) (there is a timelike, \(\xi^a\), and a spacelike, \(\eta^a\), killing field)

\[
ds^2 = -e^{2\nu}dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega dt)^2 + e^{2(\zeta - \nu)} (dr^2 + r^2 d\theta^2),
\]

Field equations in the frame of the ZAMO:

\[
D \cdot (B D \nu) = \frac{1}{2} r^2 \sin^2 \theta B^3 e^{-4\nu} D \omega \cdot D \omega + 4\pi B e^{2\zeta - 2\nu} \left[ \frac{(\epsilon + p)(1 + u^2)}{1 - u^2} + 2p \right],
\]

\[
D \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu} D \omega) = -16\pi r \sin \theta B^2 e^{2\zeta - 4\nu} \frac{(\epsilon + p)u}{1 - u^2},
\]

\[
D \cdot (r \sin \theta D B) = 16\pi r \sin \theta B e^{2\zeta - 2\nu} p,
\]

Asymptotic expansion of the metric functions:

\[
\nu = \sum_{l=0}^{\infty} \nu_{2l}(r) P_{2l}(\mu),
\]

\[
\omega = \sum_{l=0}^{\infty} \omega_{2l}(r) P_{2l+1,\mu}(\mu),
\]

\[
B = \sum_{l=0}^{\infty} B_{2l}(r) T_{2l}^{1/2}(\mu),
\]

\(^{(9)}\]

\(^{(10)}\]

\(^{(11)}\]

\(P_l\) are the Legendre polynomials, \(\mu = \cos \theta\), and \(T_l^{1/2}\) are the Gegenbauer polynomials.
From the field equations we have the asymptotic expansion:

\[
\nu = \left\{ \frac{-M}{r} + \frac{1}{3} \tilde{B}_0 \frac{M}{r^3} + \frac{J^2}{r^4} + \left[ -\frac{\tilde{B}_0^2}{5} + \frac{\tilde{B}_2}{15} - \frac{12J^2}{5} \right] \frac{M}{r^5} + \ldots \right\} \\
+ \left\{ \frac{\tilde{\nu}_2}{r^3} - 2 \frac{J^2}{r^4} + \left[ \ldots \right] \frac{1}{r^5} + \ldots \right\} P_2(\mu) \\
+ \left\{ \frac{\tilde{\nu}_4}{r^5} + \ldots \right\} P_4(\mu) + \ldots \tag{12}
\]

\[
\omega = \left\{ \frac{2J}{r^3} - \frac{6JM}{r^4} + \frac{6}{5} \left[ 8 - 3 \frac{\tilde{B}_0}{M^2} \right] \frac{JM^2}{r^5} + (\ldots) \frac{J}{r^6} + \ldots \right\} P_{1,\mu}(\mu) \\
+ \left\{ \frac{\tilde{\omega}_2}{r^5} + (\ldots) \frac{1}{r^6} - \ldots \right\} P_{3,\mu}(\mu) + \ldots \tag{13}
\]

\[
B = \left( \frac{\pi}{2} \right)^{1/2} \left( 1 + \frac{\tilde{B}_0}{r^2} \right) T_0^{1/2}(\mu) + \left( \frac{\pi}{2} \right)^{1/2} \frac{\tilde{B}_2}{r^4} T_2^{1/2}(\mu) + \ldots \tag{14}
\]
In 1989 Komatsu, Eriguchi, and Hechisu\textsuperscript{2} wrote the same line element,

\[ ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2 \theta e^{2\beta} (d\phi - \omega dt)^2 + e^{2\alpha} (dr^2 + r^2 d\theta^2). \]

where,

\[ \nu_{\text{BI}} = \nu_{\text{KEH}} = \nu, \quad B_{\text{BI}} e^{-\nu} = e^{\beta_{\text{KEH}}}, \quad \zeta_{\text{BI}} = \nu + \alpha_{\text{KEH}}, \quad (15) \]

and proposed a different scheme for integrating the field equations using Green’s functions.

The combinations of $\nu_{\text{KEH}}$ and $\beta_{\text{KEH}}$ along with $\omega$ could be expressed as power series in $1/r$:

$$\gamma = \nu_{\text{KEH}} + \beta_{\text{KEH}}, \quad \rho = \nu_{\text{KEH}} - \beta_{\text{KEH}},$$

$$\omega = \sum_{n=1}^{\infty} \frac{(-2n-1)^{2n+1}}{r^{2n+1}} + \text{higher order} P_{2n}(\mu),$$

$$\gamma = \sum_{n=1}^{\infty} \frac{2(2n-1)^{2n-1}}{r^{2n}} + \text{higher order} \frac{P_{2n-1}(\mu)}{\sin \theta} \sin(2n-1)\theta.$$
Multipole moments of numerical spacetimes.
In 1995 Ryan proposed a way of measuring the multipole moments of a spacetime from gravitational waves\(^3\).

Ryan’s procedure expressed the energy loss per logarithmic frequency interval, \(\Delta \tilde{E}\) of an inspiralling test particle as a series expansion in the orbital frequency of the circular equatorial orbits.

The coefficients of the expansion are functions of the multipole moments.

\[
\Delta \tilde{E} = \frac{1}{3} v^2 - \frac{1}{2} v^4 + \frac{20}{9} \frac{S_1}{M^2} v^5 + \left( -\frac{27}{8} + \frac{M_2}{M^3} \right) v^6 + \frac{28}{3} \frac{S_1}{M^2} v^7 \\
+ \left( -\frac{225}{16} + \frac{80}{27} \frac{S_1^2}{M^4} + \frac{70}{9} \frac{M_2}{M^3} \right) v^8 + \left( \frac{81}{2} \frac{S_1}{M^2} + 6 \frac{S_1 M_2}{M^5} - 6 \frac{S_3}{M^4} \right) v^9 + \ldots
\]

where \(v = (M \Omega)^{1/3}\) and \(S_1 = J\).

Following Ryan's procedure for the Butterworth and Ipser metric, we first express the orbital frequency

\[ \Omega = \frac{d\phi}{dt} = -g_{t\phi,r} + \frac{\sqrt{(g_{t\phi,r})^2 - g_{tt,r}g_{\phi\phi,r}}}{g_{\phi\phi,r}}, \]

as a power series in \( x = (M/r)^{1/2} \) and then invert it to obtain

\[ x = v + \frac{v^3}{2} + \frac{ju^4}{3} + \frac{1}{24}(13 + 4b - 6q)v^5 + \frac{ju^6}{2} \]
\[ + \frac{(97 + 28b + 56j^2 - 144q)v^7}{144} \]
\[ + \frac{(373j + 292bj - 330jq - 270w_2)v^8}{360} + O(v^9), \quad (20) \]

where \( j = \frac{J}{M^2}, \ q = \frac{\nu_2}{M^3}, \ w_2 = \frac{\omega_2}{M^2}, \ b = \frac{B_2}{M^2}. \)
We then expand

\[
\tilde{E} = \frac{-g_{tt} - g_{t\phi} \Omega}{\sqrt{-g_{tt} - 2g_{t\phi} \Omega - g_{\phi\phi} \Omega^2}}, \tag{21}
\]
as a power series in \( x = (M/r)^{1/2} \) and substitute the previous expression for \( x \). Then

\[
\Delta \tilde{E} = -\frac{d \tilde{E}}{d \log \Omega} = -\frac{v d \tilde{E}}{3 dv}, \tag{22}
\]
will give,

\[
\Delta \tilde{E} = \frac{v^2}{3} - \frac{v^4}{2} + \frac{20jv^5}{9} - \frac{(89 + 32b + 24q)}{24} v^6 + \frac{28jv^7}{3} - \frac{5 (1439 + 896b - 256j^2 + 672q)}{432} v^8 + \frac{((421 + 64b - 60q)j - 90w_2)}{10} v^9 + O(v^{10}), \tag{23}
\]
By equating the coefficients of the previous power series with the corresponding ones of Ryan’s expression we have \(^4\),

\[
\begin{align*}
M^\text{corr}_2 &= -\tilde{\nu}_2 - \frac{4}{3} \left( \frac{1}{4} + b \right) M^3 = M_2 - \frac{4}{3} \left( \frac{1}{4} + b \right) M^3, \\
S^\text{corr}_3 &= \frac{3}{2} \tilde{\omega}_2 - \frac{12}{5} \left( \frac{1}{4} + b \right) jM^4 = S_3 - \frac{12}{5} \left( \frac{1}{4} + b \right) jM^4.
\end{align*}
\] (24)

\[
\begin{align*}
M^\text{corr}_2 &= -\tilde{\nu}_2 - \frac{4}{3} \left( \frac{1}{4} + b \right) M^3 = M_2 - \frac{4}{3} \left( \frac{1}{4} + b \right) M^3, \\
S^\text{corr}_3 &= \frac{3}{2} \tilde{\omega}_2 - \frac{12}{5} \left( \frac{1}{4} + b \right) jM^4 = S_3 - \frac{12}{5} \left( \frac{1}{4} + b \right) jM^4.
\end{align*}
\] (25)

Consequences of correcting the multipole moments.
Quantitative difference between the corrected and the uncorrected moments.

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EOS AU.
# EOS FPS.

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### EOS L.

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Relating the higher moments of a compact object with its spin.

Plot of the the parameter $b$ and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit for sequence (iii).

$$b = -1/4 + a_1 j^2, \quad a_1 > 0.$$
Plot of the reduced quadrupole $q$ and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit with one parameter for sequence (iii), and cyan is the fit with two parameters for the same sequence.

$$q = a_2 j^2, \quad a_2 < 0.$$
Plot of the reduced spin octupole $s_3$ and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit with one parameter for sequence (iii), and cyan is the fit with two parameters for the same sequence.

$$s_3 = a_3 j^3, \quad a_3 < 0.$$
Constructing analytic vacuum solutions of Einstein’s equations to describe the spacetime exterior to neutron stars.

Using the three parameter Manko et al. analytic solution\(^5\) to check the improvement.

A typical log-log plot of the relative difference between the numerical and the analytic metric \((g^n_{ij} - g^a_{ij})/g^n_{ij}\) for a specific numerical model before (dashed curve) and after the correction of \(M_2\) (dashed-dotted curve). The left plot is for \(g_{tt}\) and the right one for \(g_{t\phi}\).

Using the four parameter Two-Soliton analytic solution\(^6\) to check the improvement.

\[
\sigma_{ij} = \left( \int_{R_s}^{\infty} (g_{ij}^n - g_{ij}^a)^2 dr \right)^{1/2}.
\]

On the left the contour plots of the overall difference for the \(g_{tt}\) and \(g_{t\phi}\) components of the numerical metric compared to the analytic Two-Soliton metric. On the right the contour plot of the overall difference for the \(\Omega\). In the middle the contour plot of the relative difference in \(R_{isco}\).

Contour plots of the overall difference for the $g_{tt}$ and $g_{t\phi}$ components of the numerical metric compared to the analytic Two-Soliton metric. On the left we have model #15 of ESO AU. In the middle we have model #28 of EOS AU. On the right we have model #28 of EOS L.
Thank you!!