Revising the Multipole Moments of numerical spacetimes.

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- A few words on multipole moments.

- Numerical spacetimes for rapidly rotating neutron stars.

- Multipole moments of numerical spacetimes.

- Consequences of correcting the multipole moments.

A few words on multipole moments.

Why are we interested in multipole moments?

- Global properties of the spacetime (the multipole structure determines the geometry),

- Relevant in gravitational wave astronomy (the observables can be related to the moments of the source),

- Constraining the equation of state for NS (from the relation between M, J, $Q, \ldots$),

- Can be used to construct analytic spacetimes for the exterior of neutron stars.

Newtonian multipole moments:

$$\Phi(r) = \frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}_a x^a}{r^3} + \frac{\mathcal{Q}_{ab} x^a x^b}{r^5} + \ldots \qquad (1)$$

where, $\mathcal{Q}$, $\mathcal{Q}_a$, $\mathcal{Q}_{ab}$, are some integrals on the source

$$\mathcal{Q} = \int \rho d^3x,\ \mathcal{Q}_a = \int x^a \rho d^3x,\ \ldots \qquad (2)$$

The multipole moments are generally tensorial quantities.

Definition of the moments at infinity:

$x^a \rightarrow \tilde{x}^a = r^{-2} x^a$: $\tilde{r}^2 = \tilde{x}^a \tilde{x}_a = r^{-2}$

$$\Phi(r) = \tilde{r} \left(\mathcal{Q} + \mathcal{Q}_a \tilde{x}^a + \mathcal{Q}_{ab} \tilde{x}^a \tilde{x}^b + ... \right) \tag{3}$$

If we define the potential at infinity $\tilde{\Phi} = \tilde{r}^{-1} \Phi$ then the moments are

$$P_{a_1 ... a_n} = \tilde{D}_{a_n} P_{a_1 ... a_{n-1}} = \tilde{D}_{a_1} ... \tilde{D}_{a_n} \tilde{\Phi} \tag{4}$$

Relativistic multipole moments:

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential" by a recursive relation,
- There are two sets of moments, the Mass moments and the Rotation moments,
- For the two sets of moments we have two generating potentials, Φ_M, Φ_J,

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.

Numerical spacetimes for rapidly rotating neutron stars.

In 1976 Butterworth and Ipser wrote the line element for a stationary and axially symmetric spacetime[1] (there is a timelike, ξ^a, and a spacelike, η^a, killing field)

$$\begin{aligned} ds^2 &= -e^{2\nu}dt^2 + r^2\sin^2\theta B^2 e^{-2\nu}(d\phi - \omega dt)^2 \\ &\quad + e^{2(\zeta-\nu)}(dr^2 + r^2 d\theta^2), \end{aligned} \tag{5}$$

Field equations in the frame of the ZAMO:

$$\mathbf{D}\cdot(B\mathbf{D}\nu) = \frac{1}{2}r^2\sin^2\theta B^3 e^{-4\nu}\mathbf{D}\omega\cdot\mathbf{D}\omega + 4\pi B e^{2\zeta-2\nu}\left[\frac{(\epsilon+p)(1+u^2)}{1-u^2} + 2p\right], \tag{6}$$

$$\mathbf{D}\cdot(r^2\sin^2\theta B^3 e^{-4\nu}\mathbf{D}\omega) = -16\pi r\sin\theta B^2 e^{2\zeta-4\nu}\frac{(\epsilon+p)u}{1-u^2}, \tag{7}$$

$$\mathbf{D}\cdot(r\sin\theta\mathbf{D}B) = 16\pi r\sin\theta B e^{2\zeta-2\nu}p, \tag{8}$$

[1] E. M. Butterworth and J. R. Ipser, ApJ **204**, 200 (1976).

Asymptotic expansion of the metric functions:

$$\nu = \sum_{l=0}^{\infty} \nu_{2l}(r) P_{2l}(\mu), \tag{9}$$

$$\omega = \sum_{l=0}^{\infty} \omega_{2l}(r) P_{2l+1,\mu}(\mu), \tag{10}$$

$$B = \sum_{l=0}^{\infty} B_{2l}(r) T_{2l}^{1/2}(\mu), \tag{11}$$

P_l are the Legendre polynomials, $\mu = \cos\theta$, and $T_l^{1/2}$ are the Gegenbauer polynomials.

From the field equations we have the asymptotic expansion:

$$
\begin{aligned}
\nu &= \left\{-\frac{M}{r} + \frac{1}{3}\tilde{B}_0\frac{M}{r^3} + \frac{J^2}{r^4} + \left[-\frac{\tilde{B}_0^2}{5} + \frac{\tilde{B}_2^2}{15} - \frac{12J^2}{5}\right]\frac{M}{r^5} + \ldots\right\} \\
&\quad + \left\{\frac{\tilde{\nu}_2}{r^3} - 2\frac{J^2}{r^4} + [\ldots]\frac{1}{r^5} + \ldots\right\} P_2(\mu) \\
&\quad + \left\{\frac{\tilde{\nu}_4}{r^5} + \ldots\right\} P_4(\mu) + \ldots \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\omega &= \left\{\frac{2J}{r^3} - \frac{6JM}{r^4} + \frac{6}{5}\left[8 - 3\frac{\tilde{B}_0}{M^2}\right]\frac{JM^2}{r^5} + (\ldots)\frac{J}{r^6} + \ldots\right\} P_{1,\mu}(\mu) \\
&\quad + \left\{\frac{\tilde{\omega}_2}{r^5} + (\ldots)\frac{1}{r^6} - \ldots\right\} P_{3,\mu}(\mu) + \ldots \qquad (13)
\end{aligned}
$$

$$
B = \left(\frac{\pi}{2}\right)^{1/2}\left(1 + \frac{\tilde{B}_0}{r^2}\right) T_0^{1/2}(\mu) + \left(\frac{\pi}{2}\right)^{1/2}\frac{\tilde{B}_2}{r^4} T_2^{1/2}(\mu) + \ldots \qquad (14)
$$

In 1989 Komatsu, Eriguchi, and Hechisu[2] wrote the same line element,

$$ds^2 = -e^{2\nu}dt^2 + r^2 \sin^2\theta e^{2\beta}(d\phi - \omega dt)^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2).$$

where,

$$\nu_{\rm BI} = \nu_{\rm KEH} = \nu,\ B_{\rm BI}e^{-\nu} = e^{\beta_{\rm KEH}}\ ,\ \zeta_{\rm BI} = \nu + \alpha_{\rm KEH}, \tag{15}$$

and proposed a different scheme for integrating the field equations using Green's functions.

[2]H. Komatsu, Y. Eriguchi, and I. Hechisu, MNRAS **237**, 355 (1989).

The combinations of ν_{KEH} and β_{KEH}

$$\gamma = \nu_{\text{KEH}} + \beta_{\text{KEH}} \ , \ \rho = \nu_{\text{KEH}} - \beta_{\text{KEH}}, \tag{16}$$

along with ω could be expressed as power series in $1/r$:

$$\rho = \sum_{n=0}^{\infty} (-2\frac{M_{2n}}{r^{2n+1}} + \text{higher order}) P_{2n}(\mu), \tag{17}$$

$$\omega = \sum_{n=1}^{\infty} (-\frac{2}{2n-1}\frac{S_{2n-1}}{r^{2n+1}} + \text{higher order}) \frac{P^1_{2n-1}(\mu)}{\sin\theta}, \tag{18}$$

$$\gamma = \sum_{n=1}^{\infty} (\frac{D_{2n-1}}{r^{2n}} + \text{higher order}) \frac{\sin(2n-1)\theta}{\sin\theta}. \tag{19}$$

The coefficients M_{2n} and S_{2n-1} were identified as the mass and current-mass moments, respectively. By comparison between the above expansion and the corresponding ones of BI we see that $M_2 = -\tilde{\nu}_2$ and $S_3 = \frac{3}{2}\tilde{\omega}_2$.

Multipole moments of numerical spacetimes.

In 1995 Ryan proposed a way of measuring the multipole moments of a spacetime from gravitational waves[3].

Ryan's procedure expressed the energy loss per logarithmic frequency interval, $\Delta\tilde{E}$ of an inspiralling test particle as a series expansion in the orbital frequency of the circular equatorial orbits.

The coefficients of the expansion are functions of the multipole moments.

$$\begin{aligned}\Delta\tilde{E} \;=\; & \frac{1}{3}\upsilon^2 - \frac{1}{2}\upsilon^4 + \frac{20}{9}\frac{S_1}{M^2}\upsilon^5 + \left(-\frac{27}{8} + \frac{M_2}{M^3}\right)\upsilon^6 + \frac{28}{3}\frac{S_1}{M^2}\upsilon^7 \\ & + \left(-\frac{225}{16} + \frac{80}{27}\frac{S_1^2}{M^4} + \frac{70}{9}\frac{M_2}{M^3}\right)\upsilon^8 + \left(\frac{81}{2}\frac{S_1}{M^2} + 6\frac{S_1 M_2}{M^5} - 6\frac{S_3}{M^4}\right)\upsilon^9 + \ldots\end{aligned}$$

where $\upsilon = (M\Omega)^{1/3}$ and $S_1 = J$.

[3] F. D. Ryan, Phys. Rev. D **52**, 5707 (1995).

Following Ryan's procedure for the Butterworth and Ipser metric, we first express the orbital frequency

$$\Omega = \frac{d\phi}{dt} = \frac{-g_{t\phi,r} + \sqrt{(g_{t\phi,r})^2 - g_{tt,r}g_{\phi\phi,r}}}{g_{\phi\phi,r}},$$

as a power series in $x = (M/r)^{1/2}$ and then invert it to obtain

$$\begin{aligned} x &= v + \frac{v^3}{2} + \frac{jv^4}{3} + \frac{1}{24}(13 + 4b - 6q)v^5 + \frac{jv^6}{2} \\ &+ \frac{(97 + 28b + 56j^2 - 144q)v^7}{144} \\ &+ \frac{(373j + 292bj - 330jq - 270w_2)v^8}{360} + O(v^9), \end{aligned} \tag{20}$$

where $j = \frac{J}{M^2},\ q = \frac{\nu_2}{M^3},\ w_2 = \frac{\omega_2}{M^4},\ b = \frac{B_0}{M^2}$.

We then expand

$$\tilde{E} = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \tag{21}$$

as a power series in $x = (M/r)^{1/2}$ and substitute the previous expression for x. Then

$$\Delta\tilde{E} = -\frac{d\tilde{E}}{d\log\Omega} = -\frac{v}{3}\frac{d\tilde{E}}{dv}, \tag{22}$$

will give,

$$\begin{aligned}\Delta\tilde{E} \;=\; & \frac{v^2}{3} - \frac{v^4}{2} + \frac{20jv^5}{9} - \frac{(89+32b+24q)}{24}v^6 + \frac{28jv^7}{3} \\ & - \frac{5\left(1439+896b-256j^2+672q\right)v^8}{432} \\ & + \frac{((421+64b-60q)j-90w_2)\,v^9}{10} + O(v^{10}),\end{aligned} \tag{23}$$

By equating the coefficients of the previous power series with the corresponding ones of Ryan's expression we have [4],

$$M_2^{corr} = -\tilde{\nu}_2 - \frac{4}{3}(\frac{1}{4}+b)M^3 = M_2 - \frac{4}{3}\left(\frac{1}{4}+b\right)M^3, \quad (24)$$

$$S_3^{corr} = \frac{3}{2}\tilde{\omega}_2 - \frac{12}{5}(\frac{1}{4}+b)jM^4 = S_3 - \frac{12}{5}\left(\frac{1}{4}+b\right)jM^4. \quad (25)$$

[4] G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** (2012), 231104.

Consequences of correcting the multipole moments.

Quantitative difference between the corrected and the uncorrected moments.

EOS AU.

#	M	j	M_2	S_3	b	$\Delta M_2(\%)$	$\Delta S_3(\%)$
4	2.087	0.414	-6.08	-10.0	-0.23	4.032	3.805
9	2.111	0.650	-15.2	-40.5	-0.20	4.008	3.716
10	2.112	0.661	-15.7	-42.7	-0.20	4.002	3.706
12	3.164	0.194	-1.68	-1.37	-0.24	21.63	31.69
15	3.231	0.485	-12.0	-26.6	-0.20	18.43	24.82
17	3.273	0.603	-20.2	-58.1	-0.18	16.92	21.80
19	3.304	0.676	-27.0	-90.4	-0.17	15.86	19.77
20	3.318	0.706	-30.3	-107.	-0.16	15.40	18.88
21	3.388	0.510	-12.8	-27.8	-0.19	26.05	42.10
22	3.388	0.510	-12.9	-28.1	-0.19	25.75	41.32
28	3.458	0.659	-26.0	-82.7	-0.17	19.79	27.20
29	3.477	0.694	-30.1	-103.	-0.16	18.91	25.31
30	3.487	0.713	-32.5	-115.	-0.16	18.45	24.34

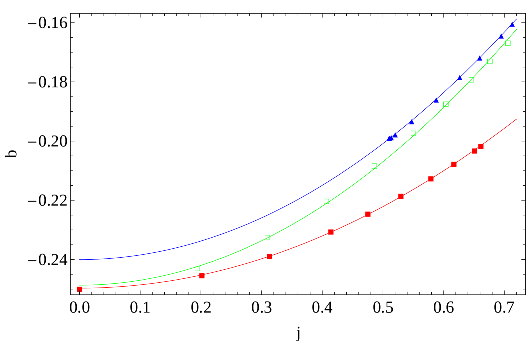
EOS FPS.

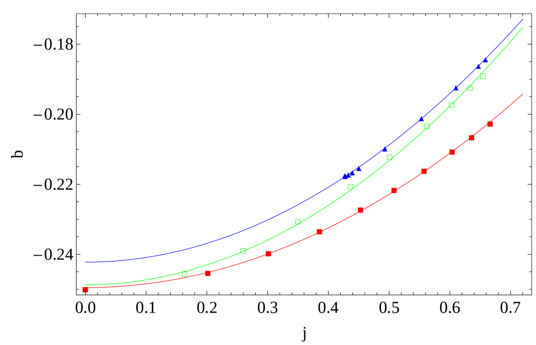
#	M	j	M_2	S_3	b	$\Delta M_2(\%)$	$\Delta S_3(\%)$
5	2.087	0.452	-7.85	-14.6	-0.22	3.634	3.311
7	2.098	0.557	-12.0	-28.0	-0.21	3.575	3.228
9	2.106	0.636	-15.9	-42.9	-0.20	3.504	3.136
10	2.109	0.666	-17.6	-49.9	-0.20	3.469	3.091
14	2.686	0.349	-4.81	-7.43	-0.23	11.63	12.90
17	2.727	0.562	-14.6	-39.0	-0.20	9.428	9.805
19	2.744	0.633	-19.7	-60.8	-0.19	8.734	8.866
20	2.750	0.654	-21.4	-69.0	-0.18	8.543	8.607
21	2.823	0.427	-6.54	-11.7	-0.21	17.55	22.14
22	2.823	0.428	-6.69	-12.1	-0.21	16.97	21.12
29	2.882	0.647	-21.1	-67.6	-0.18	10.64	11.25
30	2.884	0.658	-22.2	-72.8	-0.18	10.42	10.94

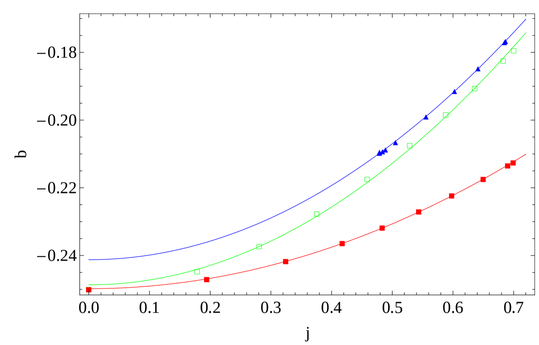
EOS L.

#	M	j	M_2	S_3	b	$\Delta M_2(\%)$	$\Delta S_3(\%)$
4	2.080	0.417	-12.2	-22.0	-0.23	1.352	1.175
7	2.090	0.598	-24.4	-63.5	-0.22	1.399	1.210
9	2.096	0.690	-32.1	-97.1	-0.21	1.417	1.220
10	2.097	0.698	-32.9	-100.	-0.21	1.420	1.222
14	4.051	0.375	-18.5	-45.4	-0.22	11.94	13.57
17	4.120	0.588	-51.8	-210.	-0.19	10.25	11.09
19	4.160	0.682	-74.9	-365.	-0.18	9.478	9.998
20	4.167	0.700	-79.8	-401.	-0.17	9.335	9.794
21	4.321	0.478	-29.0	-88.0	-0.20	17.68	22.66
22	4.321	0.479	-29.5	-90.2	-0.20	17.19	21.82
28	4.396	0.641	-66.0	-299.	-0.18	12.60	14.32
29	4.418	0.684	-78.7	-389.	-0.17	11.92	13.28
30	4.420	0.686	-79.4	-394.	-0.17	11.89	13.24

Relating the higher moments of a compact object with its spin.

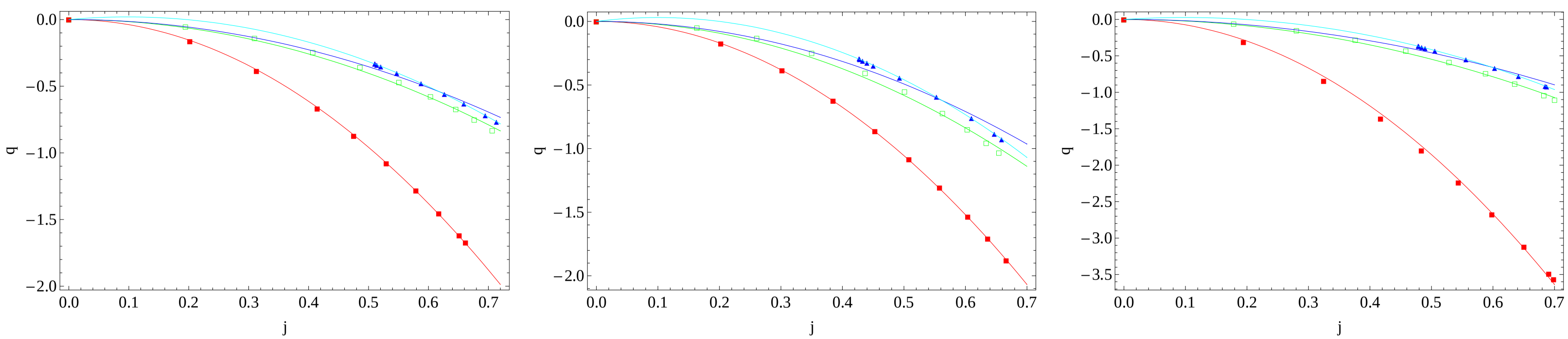






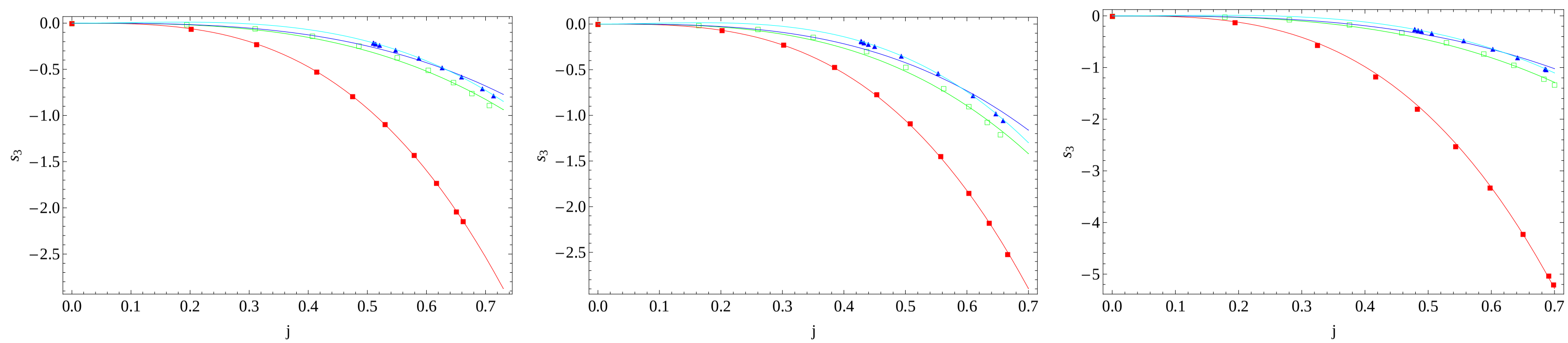
Plot of the the parameter b and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit for sequence (iii).

$$b = -1/4 + a_1 j^2, \quad a_1 > 0.$$



Plot of the reduced quadrupole q and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit with one parameter for sequence (iii), and cyan is the fit with two parameters for the same sequence.

$$q = a_2 j^2, \quad a_2 < 0.$$

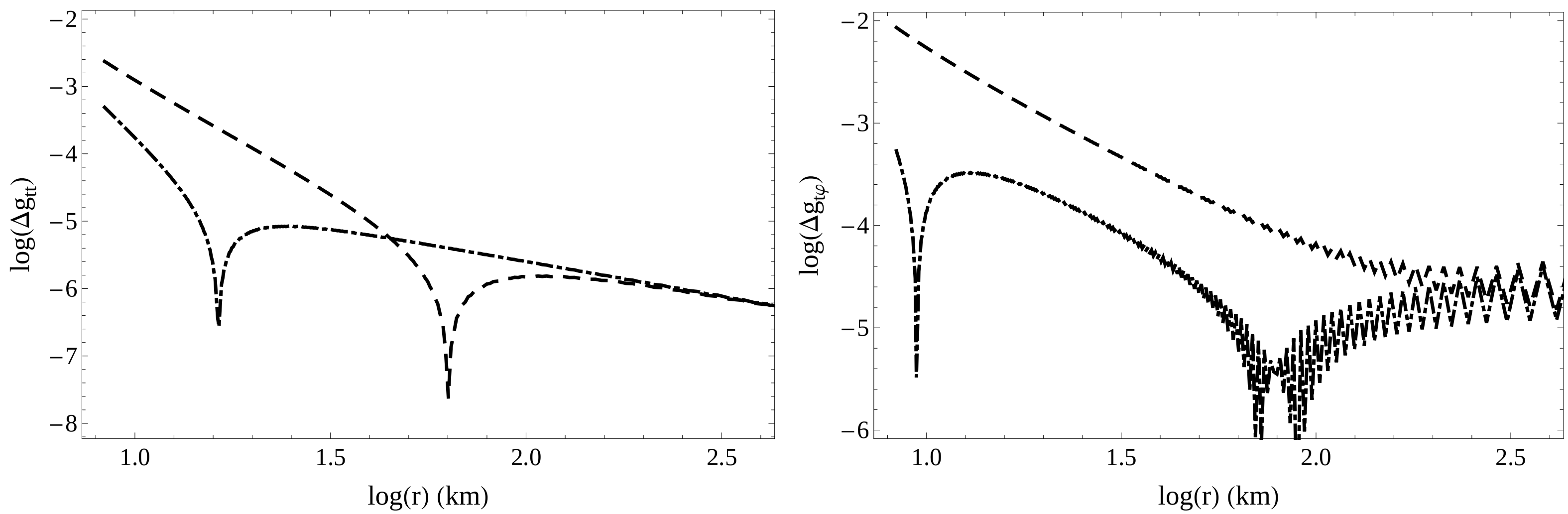


Plot of the reduced spin octupole s_3 and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit with one parameter for sequence (iii), and cyan is the fit with two parameters for the same sequence.

$$s_3 = a_3 j^3, \quad a_3 < 0.$$

Constructing analytic vacuum solutions of Einstein's equations to describe the spacetime exterior to neutron stars.

Using the three parameter Manko et al. analytic solution[5] to check the improvement.

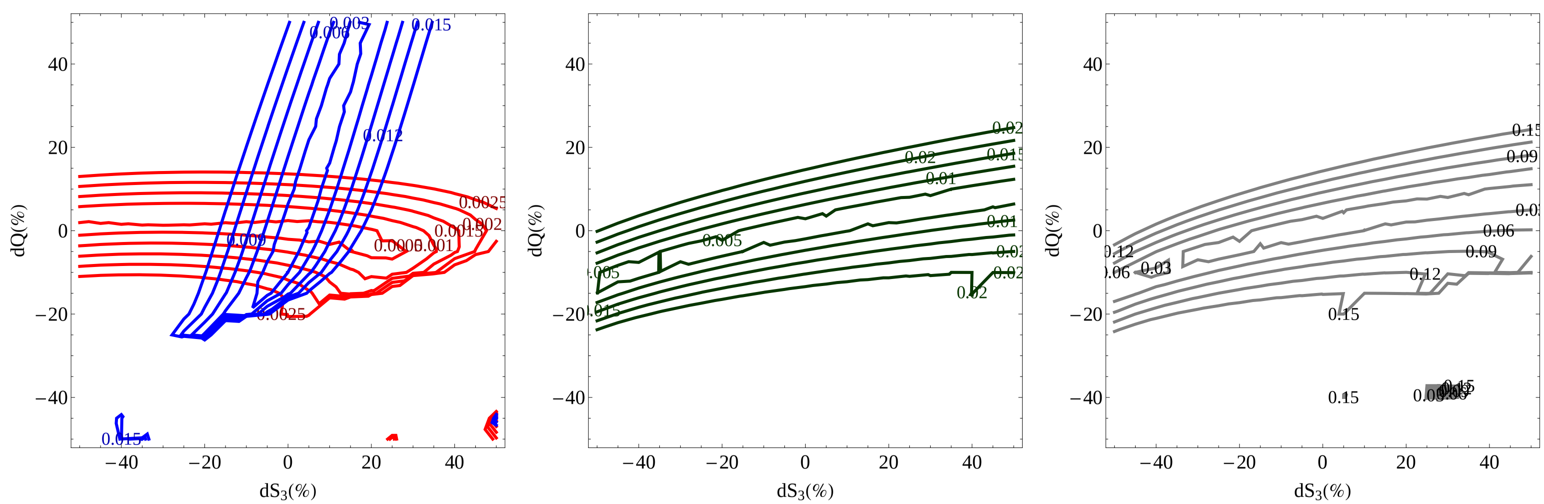


A typical log-log plot of the relative difference between the numerical and the analytic metric ($(g^n_{ij} - g^a_{ij})/g^n_{ij}$) for a specific numerical model before (dashed curve) and after the correction of M_2 (dashed-dotted curve). The left plot is for g_{tt} and the right one for $g_{t\phi}$.

[5]V. S. Manko, E. W. Mielke, and J. D. Sanabria-Gómez, Phys. Rev. D **61**, 081501 (2000)

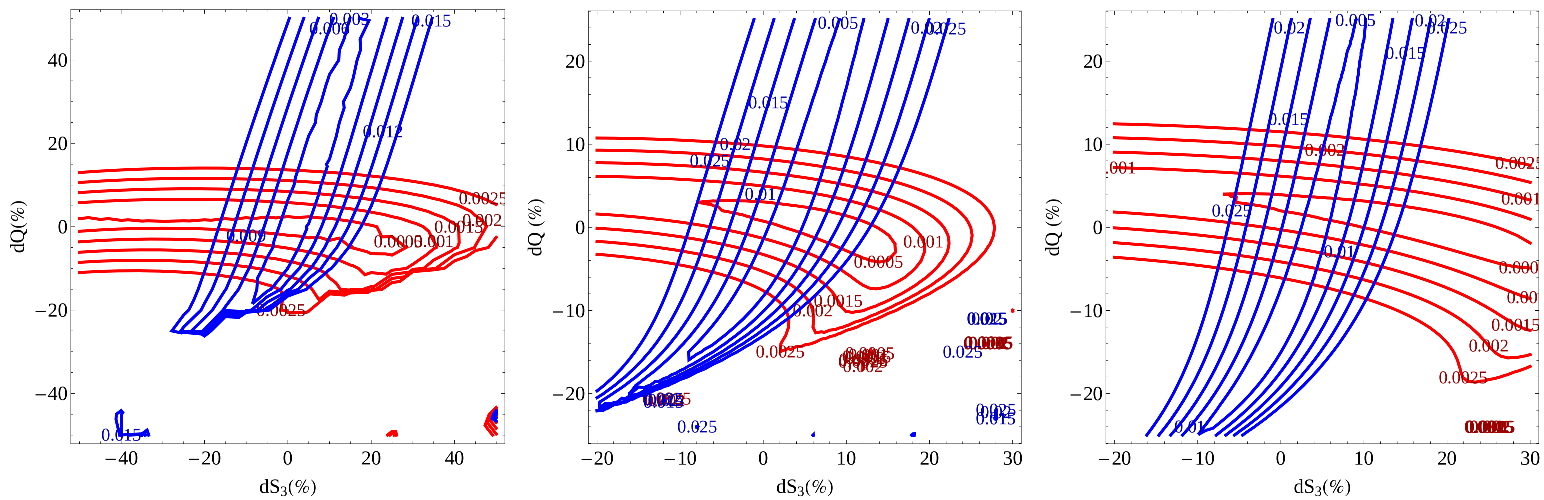
Using the four parameter Two-Soliton analytic solution[6] to check the improvement.

$$\sigma_{ij} = \left(\int_{R_S}^{\infty} (g_{ij}^n - g_{ij}^a)^2 dr\right)^{1/2}.$$



On the left the contour plots of the overall difference for the g_{tt} and $g_{t\phi}$ components of the numerical metric compared to the analytic Two-Soliton metric. On the right the contour plot of the overall difference for the Ω. In the middle the contour plot of the relative difference in R_{isco}.

[6]V. S. Manko, J. Martin, E. Ruiz, 1995, *J. Math. Phys.* **36** 3063



Contour plots of the overall difference for the g_{tt} and $g_{t\phi}$ components of the numerical metric compared to the analytic Two-Soliton metric. On the left we have model #15 of ESO AU. In the middle we have model #28 of EOS AU. On the right we have model #28 of EOS L.

Thank you!!!