Local existence for Einstein-Euler equations

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Fluids filling riemannian manifolds

We will study fluids in the context of general relativity and derive certain local elliptic type estimates for the fluid quantities. The main motivation for these studies are numerical codes for fluids.

Basic equations

Metric of the form

$$g = -\varphi^2 dt^2 + \gamma(t)$$

The slices

$$\mathcal{H}_t = \{x \in M / t(x) = t\}$$

are spacelike hypersurfaces that carry the metric γ. We will assume that the initial slice $\mathcal{H}_0$ is a 3-manifold equipped with smooth initial data φ, γ, k, and contains a fluid with data $(\underline{v}, \Omega, \theta, \widetilde{p}), \widetilde{p} = \varrho + p = e^w)$.

Einstein equations read

$$\mathbf{R}_{\mu\nu} = 2\left(\mathbf{T}_{\mu\nu} - \frac{1}{2}\mathbf{T}g_{\mu\nu}\right)$$

Energy momentum tensor of a perfect fluid is

$$\mathbf{T}_{\mu\nu} = \widetilde{p}v_\mu v_\nu + pg_{\mu\nu}, \quad \widetilde{p} = \varrho + p$$

The contracted second Bianchi identities demand that the energy-momentum tensor satisfies the continuity equation

$$\nabla^\nu \mathbf{T}_{\mu\nu} = 0$$

In the case of the perfect fluid these comprise the continuity and the Euler's equation.

The fluid velocity is chosen as $g(\mathbf{v}, \mathbf{v}) = -1$ which gives that for $\mathbf{v} = (v^0, \underline{v})$:

$$\varphi^2 (v^0)^2 = 1 + ||\underline{v}||_\gamma^2 =: \tau^2$$

Notice that

$$\mathbf{T}_{00} = (\widetilde{p}\tau^2 - p)\varphi^2 \geq (\widetilde{p}|\underline{v}|_\gamma^2 + \varrho)\varphi^2 \tag{1}$$

$$\mathbf{T} = 3p - \varrho$$

The induced riemannian metric on the slices $\mathcal{H}_t$ is denoted by $\gamma(t)$ and the second fundamental form is $k(t)$ and we have the first variation identity

$$\partial_t \gamma = -2\varphi k \quad \text{(FV)}$$

The second variation formula and the Gauss-Codazzi (GC) and Gauss equations under the assumption of maximal slicing (locally exists) i.e. $\mathrm{tr}(k) = 0$ are written as:

$$\partial_t k_{ij} = -\nabla_i \nabla_j \varphi + \left(\mathrm{R}_{ij} - 2k_{im}k_j^m - \frac{1}{2}\mathrm{R}\gamma_{ij} - 2\mathbf{T}_{ij} \right) \varphi \quad \text{(SV)}$$

$$\nabla_i k_{jm} - \nabla_j k_{im} = \mathbf{R}_{m0ij} \quad \text{(GC)}$$

$$\mathrm{R}_{ij} - k_{il}k_j^l = \mathbf{R}_{i0j0} + \mathbf{R}_{ij} \quad \text{(G)}$$

Taking the trace of (G) we obtain that:

$$\mathrm{R} - |k|^2 = \mathbf{T}_{00} \Rightarrow |k|^2 + \widetilde{p}\tau^2\varphi^2 = \varphi^2 p + \mathrm{R}$$

We observe that

$$|k|^2 + p|\underline{v}|^2 + \rho\tau = \mathrm{R} \Rightarrow \mathrm{R} \geq p|\underline{v}|^2, |k|^2, \rho\tau$$

The trace of (GC) gives us that:

$$\mathrm{div}(k) = 2T_{j0} = 2\widetilde{p}\tau v_j$$

Furthermore the trace of (SV) we obtain that

$$\varphi\Delta_\gamma\varphi + \left(|k|^2 + 2(\mathbf{T}_{00} + \frac{1}{2}\varphi^2\mathbf{T})\right)\varphi^2 = 0$$

hence the differential equation for $\delta = p - \rho$:

$$\varphi\Delta_\gamma\varphi = -\left(2\pi\tau^2 + |k|^2 + \delta\varphi^2\right)\varphi^2 \quad \text{(L)}$$

Wave equations for the second fundamental form

Moreover the second variation equation provides through the equation for the Ricci curvtaure the wave equation for the second fundamental form;

$$\varphi \Box_g k = \mathscr{S}$$

where

$$\mathscr{S}_{ij} = \mathscr{S}_{00,ij} + \mathscr{S}_{01,ij} + \mathscr{S}_{10,ij} + \mathscr{S}_{11,ij} + \mathscr{S}_{2,ij} + \mathscr{S}_{3,ij}$$

$$
\begin{aligned}
\mathscr{S}_{00,ij} &= -4\varphi\left[k_i{}^l k_l{}^m k_{jm} - \frac{1}{2}\mathrm{R}^l{}_{ijm}k^m_l - \frac{3}{4}(\mathrm{R}_{mi}k^m_j + \mathrm{R}_{jm}k^m_i) + \frac{1}{2}\mathrm{R}k_{ij}\right] \\
\mathscr{S}_{01,ij} &= -4\varphi\left[\widetilde{p}(k_{il}v^l v_j + k_{jl}v^l v_i) + p\left(\gamma_{lj}k_i{}^l + \gamma_{li}k_j{}^j\right)\right] \\
\mathscr{S}_{10,ij} &= 2(\nabla_l\varphi)(\nabla_i k_j{}^l + \nabla_j k_i{}^l) - 3\nabla_l\varphi\nabla^l k_{ij} + \varphi^{-1}k_{ij}|\nabla\varphi|^2 \\
\mathscr{S}_{11,ij} &= \varphi^{-1}\left(k_{li}\nabla_j\varphi + k_{lj}\nabla_i\varphi\right)\nabla^l\varphi \\
\mathscr{S}_{2,ij} &= -\left(k_{jr}\nabla_i\nabla^r\varphi + k_{ir}\nabla_j\nabla^r\varphi + k_{ij}\Delta_\gamma\varphi\right) \\
\mathscr{S}_{3,ij} &= \partial_t\mathbf{T}_{ij} + \nabla_i(\varphi^2\mathbf{T}_j{}^0) + \nabla_j(\varphi^2\mathbf{T}_i{}^0)
\end{aligned} \tag{2}
$$

Recall that the wave operator is written as

$$
\Box_g = -\varphi^{-2}\partial_t^2 + \Delta_\gamma
$$

We assume that the fluid satisfies a barotropic equation of state:

$$p = f(\rho, S) \equiv f(\rho)$$

The sound speed in the fluid is denoted by

$$\eta^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = f'(\rho), \quad w = \log \widetilde{p}$$

and we also introduce the quantity

$$\sigma = \left(\frac{\partial^2 f}{\partial \rho^2}\right)_S = f''$$

Fluid equations read

$$\left(v^\nu v^\mu + \frac{\eta^2}{1+\eta^2} g^{\nu\mu}\right) \nabla_\mu w = -\theta v_\nu - v^\mu \nabla_\mu v_\nu$$

Acoustical metric

We introduce the *acoustical metric*:

$$\widetilde{g} = \frac{\eta^2}{1+\eta^2} g^{\nu\mu} + v^\nu v^\mu$$

with determinant $-d$

$$d = \frac{\eta^6}{(1+\eta^2)^4}$$

We form at the wave operator for $\widetilde{g}$ and obtain the wave equation for w:

$$\square_{\widetilde{g}} w = -\theta^2 - 2v^\nu \nabla_\nu \theta - \mathrm{Ric}(v, v) - \nabla_\nu v^\mu \nabla_\mu v^\nu$$

Expansion

$$\theta := \nabla_\mu v^\mu$$

Velocity gradient

$$\nabla_\mu v_\nu = \Omega_{\mu\nu} + S_{\mu\nu} + \frac{1}{3}\theta\Pi_{\mu\nu} + \frac{\eta^2}{1+\eta^2}\Pi_\mu^\kappa\nabla_\kappa\tau$$

Vorticity tensor:

$$\Omega_{\mu\nu} = \frac{1}{2}\left(\Pi^{\kappa}_{\mu}\nabla_{\kappa}u_{\nu} - \Pi^{\kappa}_{\nu}\nabla_{\kappa}v_{\mu}\right)$$

Shear tensor

$$S_{\mu\nu} = \frac{1}{2}\left[\Pi^{\kappa}_{\mu}\nabla_{\kappa}u_{\nu} + \Pi^{\kappa}_{\nu}\nabla_{\kappa}u_{\mu}\right] - \frac{1}{3}\theta\Pi_{\mu\nu}$$

$$v^\alpha \nabla_\alpha \theta = \Omega^2 - S^2 - \frac{1}{3}\theta^2 + 3r$$

$$v^\alpha \nabla_\alpha \Omega_{\mu\nu} = -\frac{2}{3}\theta \Omega_{\mu\nu} - (S_{\mu\kappa}\Omega^\kappa_\nu + \Omega_{\mu\kappa}S^\kappa_\nu)$$

$$v^\alpha \nabla_\alpha S_{\mu\nu} = -\frac{2}{3}\theta S_{\mu\nu} - (S_{\mu\kappa}S^\kappa_\nu + \Omega_{\mu\kappa}\Omega^\kappa_\nu) - \mathscr{R}_{\mu\nu} + \frac{1}{3}(S^2 - \omega^2 + r)\Pi_{\mu\nu}$$

Recall that the curvature quantities appearing above are:

$$\mathscr{R}_{00} = \mathrm{R}_{l0m0} v^l v^m, \ \mathscr{R}_{0j} = \mathrm{R}_{l0mj} v^l v^j$$

$$\mathscr{R}_{ij} = \frac{\tau^2}{\Phi^2} \mathrm{R}_{0i0j} + \frac{\tau}{\Phi} \mathrm{R}_{0ilj} v^l + \mathrm{R}_{limj} v^l v^j$$

Geodesic pixels

Points $\mathscr{C}_0 = \{\underline{C}_i^0\}_{i=1}^N \subset \mathcal{H}_t$ at distance $d(\underline{C}_i^0, \underline{C}_j^0) \sim_0$ geodesic balls of radii r_i, $B^3_{\underline{C}_i^0, r_i}, S^2_{\underline{C}_i^0, r_i}$. Overlap in quadruples

$$\mathscr{B}_{0;i_1\dots i_k} = \bigcap_{j=1}^{k} B^3_{\underline{C}^0_{i_j}, r_{i_j}} \quad k = 2, 3, 4$$

spherical regions denoted as $\mathscr{F}^{0;i_1\dots i_k}$ interior curvature data

$$\mathrm{Ric}^{0;i_1\dots i_k}$$

Face second fundamental form

$$\not{h}^{0;i_1\dots i_k}, \quad \not{k}^{0;i_1\dots i_K},$$

geodesic pixels.

Generations of such pixels after the introduction of new centers and arrive at the collection of pixels after generation j:

$$\mathscr{B}_{j;i_1\dots i_k}$$

with elementary wave fronts $\mathscr{F}^{j;i_1\dots i_k}$ and curvature data:

$$\mathrm{Ric}^{j;i_1\dots i_k}, \quad \not{h}^{j;i_1\dots i_k}, \quad \not{k}^{j;i_1\dots i_K},$$

It is written in the form for $\eta_{\ell;i_1\dots i_k}, \mu_{\ell;i_1,\dots,ik}$:

$$\mathscr{F}^{\ell;i_1\dots i_k} = \bigcup_{j=1}^{k} \mathscr{F}^{\ell;i_1\dots i_j}$$

The faces $\mathscr{F}^{\ell;i_1\ldots i_j}$ are called *elementary wave fronts* (EWF). Each pixel defines homothetic EWF spanned by the tubular neighbouhoods of the elementary wave fronts:

$$\mathscr{S}_{r,\varepsilon,\ell;i_1,\ldots,i_k} = \mathfrak{I}_{r,\varepsilon} \times \mathscr{F}^{\ell;i_1\ldots i_k} = ((1-\varepsilon)r,(1+\varepsilon)r) \times \mathscr{F}^{\ell;i_1\ldots i_k}$$

Localized tension $T(h;\vartheta)$ of an EWF given for a smooth test function ζ, supp$\zeta \subset \mathscr{F}^{\ell;i_1\ldots i_k}$:

$$T(h;\zeta) = \int_{\mathscr{F}_{k,\ell}} |\nabla\!\!\!/ h|^2\zeta^2$$

$h\!\!\!/$ mean curvature of the (EWF).

EWF satisfies an η, ϵ condition when localilized tension satisfies the estimate :

$$T(h;\varphi) \leq \eta_\ell \int_{\mathscr{F}_{k,\ell}} \zeta^2 h^4$$

Geodesic pixel satisfies the μ, ϵ condition if its Ricci curvature satisfies

$$\mathcal{E}^{(1)}(\mathrm{R};\zeta) \leq \mu \mathcal{E}^{(0)}(\mathrm{R};\zeta)$$

$$\mathcal{E}(B;\zeta) \leq \kappa \mathcal{E}^{(0)}(\mathrm{R};\zeta)$$

Bach tensor $B = \mathrm{curl}(\mathrm{Ric})$. Geodesic pixel centered at $C^{\ell}_{i_\ell}$ produced in the ℓ-th generation

$$\mathscr{B}_{\kappa_{\ell;i_\ell};\eta_{\ell;i_\ell};\mu_{\ell;i_\ell};\varepsilon_{\ell;i_\ell}}$$

are selected so that their fronts

$$\mathscr{F}^{\ell;i_1\ldots i_k}(\eta_{\ell;i_1\ldots i_k};\varepsilon_{\ell;i_1\ldots i_k})$$

satify $\eta_{\ell;i_1\ldots i_k}, \epsilon_{\ell;i_1\ldots i_k}$ estimate while its curvature satisfies a $\mu_{\ell,i_\ell}, \kappa_{\ell,i_\ell}, \epsilon_{\ell,i_\ell}$ estimate.

Geodesic coordinates from the neighboring pixels. $B^3_{\underline{C}^j_i, r_i}$ is a geodesic ball centered at the point $\underline{C}^j_i$ and introduce polar coordinates through Gauss lemma. The metric is written then as:

$$g = dr^2 + \not{\gamma}(r)$$

$\not{\gamma}(r)$ is a riemannian metric on the geodesic sphere $S^2_{\underline{C}^j_i, r_i} = \partial B^3_{\underline{C}^j_i, r_i}$ with second fundamental form and mean curvature respectively $\not{k}, \not{h}$.

The first variation system of equations for the metric on the geodesic sphere is written then explicitly as follows. Let

$$\underline{\eta} = (\not{\gamma}_{11}, \not{\gamma}_{22}, \not{\gamma}_{12}),$$

$$\underline{\xi} = (\not{k}_{11}, \not{k}_{22}, \not{k}_{12})$$

Moreover we set:

$$\not{h} = \not{\gamma}^{11}\not{k}_{11} + 2\not{\gamma}^{12}\not{k}_{12} + \not{\gamma}^{22}\not{k}_{22}$$

$$\not{v} = \not{\gamma}_{11}\not{\gamma}_{22} - \not{\gamma}_{12}^{2}, \quad \not{\kappa} = \not{k}_{11}\not{k}_{22} - \not{k}_{12}^{2}$$

$$\mu_j = \not{h}\xi_j - \frac{\not{\kappa}}{\not{v}}\eta_j, \quad j = 1, 2, 3$$

$$\underline{\epsilon} = (\mathrm{R}_{1N1N}, \mathrm{R}_{2N2N}, \mathrm{R}_{1N2N})$$

$$\frac{d\eta}{dr} = 2\underline{\xi},$$

$$\frac{d\xi}{dr} = h\underline{\xi} - \frac{\not{\kappa}}{\not{v}}\eta - \underline{\epsilon}$$

Also we have the propagation equations for the mean curvature of the front:

$$\frac{d\not{v}}{dr} = 2\not{v}\not{h},$$

and the mean curvature satisfies

$$\frac{d\not{h}}{dr} = -2\nu + \not{h}^2 - 2\frac{\not{\kappa}}{\not{v}} - \mathrm{R}_{NN}$$

where we have set the norm of the second fundamental form: $\nu = \not{k}^{ij}\not{k}_{ij}$. The latter satisfies:

$$\frac{d\nu}{dr} = 6\not{h}(\nu - \frac{\not{\kappa}}{\not{v}}) - 2\varphi$$

where

$$\varphi = \mathrm{R}^1{}_{NN1}\xi^1 + \mathrm{R}^2{}_{NN2}\xi^2 + 2\mathrm{R}^1{}_{NN2}\xi^3$$

Gauss-Codazzi

The Gauss equations that relate the curvature of $\not\gamma, \not\mathscr{R}$ to the ambient curvature

$$
\begin{aligned}
\not\mathscr{R}_{1212} + \not\kappa &= \mathrm{R}_{1212}, && (\mathrm{G}) \\
\not\mathscr{R}_{ij} + \not k_{ij} \not h - \not k_{im} \not k_j^m &= \mathrm{R}_{ij}, && (\mathrm{G}_{01}) \\
\not\mathscr{R} + \not h^2 - \not k^2 &= \mathrm{R} - \mathrm{R}_{00}, && (\mathrm{G}_{02})
\end{aligned}
$$

Radial derivative by $;0$ and anggular ones by $;j, \not\nabla$

$$\kappa_j = |\not\nabla^j \not k|, \quad \eta_j = |\not\nabla^j \not h|$$

$$\frac{d\not h}{dr} = \not k^2 - \mathrm{R}_{NN} \qquad (\mathrm{SV}_0)$$

$$\frac{d\not\nabla^j \not h}{dr} = \sum_{i_1+i_2=j} c_{i_1 i_2} \not\nabla^{i_1} \not k * \not\nabla^{i_2} \not k - \sum_{i=0}^{j-1} \mathrm{R}_{00} \not\nabla^i \not h - \not\nabla^j \mathrm{R}_{NN}$$

Similarly the Codazzi equations

$$\not\nabla_i k_{jm} - \not\nabla_j k_{im} = R_{mNij}$$

are written in the form of a 2-D Hodge system:

$$\not{\text{curl}}(k)_{ijl} = R_{lNij}, \quad \not{\text{div}}(k)_i - \not\nabla_i h = R_{Ni}$$

Approximation scheme

We will approximate the solution with the sequence

$$\varphi_n, \quad k_n, \quad g_n, \quad \widetilde{p}_n, \quad v_n$$

with given initial data

$$\varphi_0 = \varphi(0), \quad g_0 = g(0) \quad k_0 = k(0), \quad \widetilde{p}_0 = \widetilde{p}(0), \quad v_0 = v(0)$$

Solve the equations

$$\Box_{\widetilde{g}_n} \widetilde{p}_{n+1} = \mathcal{F}_n$$

$$\Delta_{g_n} \varphi_{n+1} = -\left(2\pi_n \tau_n^2 + |k_n|^2 + \delta_n \varphi_n^2\right) \varphi_{n+1}$$

$$v_n^\mu \nabla_\mu^n v_{n+1} + v_{n+1}^\mu \nabla_\mu^n v^n - \theta_n v_n - \widetilde{g}_n(\nabla w)$$

The basic tools for establishing the convergence of this scheme for time interval is provided throught the various energy conservations and and the following generalization of "Morrey lemma" which accompanied with the corresponding $L_\infty - L^p$ estimates provide the Harnack inequality

We will use repeatedly the following method that we call *harmonic approximation method*. The domains encountered here $\mathscr{W}$ have boundaries with singularities that are of a specific type: the singular set $\mathscr{S}(\partial\mathscr{W})$ is given by the transversal intersection of hypersurfaces: geodesic spheres with local equations $s_1, \ldots, s_\ell, \ell = n, n+1$. The piece of the hypersurface $\mathscr{H}_\epsilon = \{\underline{x} \in \mathscr{W}(s_1 \cdots s_\ell + \epsilon)(\underline{x} = 0\}$ near $\mathscr{S}$ is for suitable ϵ a smooth hypersurface close to $\mathscr{S}(\partial\mathscr{W})$. We will consider the domain $\tilde{\mathscr{W}}$ obtained by replacing the singular part $\mathscr{S}(\mathscr{W})$ by $\mathscr{H}_\epsilon$ with repacing the defining function through cut-offs by the function given there. Let $\widehat{F} : \tilde{\mathscr{W}} \to \mathbf{R}$ be the solution of the boundary value problem:

$$\Delta\widehat{F} = 0, \qquad \widehat{F}|_{\partial\tilde{\mathscr{W}}} = F$$

The suitable selection of the pixels will restrict the undesirable pieces do not contain nodal information: their size is so small that we could neglect them so that in first order we neglect it.

Harmonic polynomials We will also approximate the harmonic function defined in the pixel $\widehat{F}$ by a sequence $\{F_n\}_{n\in\mathbf{N}}$ of functions such that

$$\Delta_0\widehat{F_0} = 0\Delta_0\widehat{F}_n = -\sum_{i,j}\mathfrak{R}^{ij}\frac{\partial^2 F_{n-1}}{\partial x_i \partial x_j} - g^{ij}\partial_i\psi\partial_j F_{n-1} \qquad (3)$$

where

$$g_{ij} = \delta_{ij} + \mathfrak{R}_{ij}$$

and for $j = 0, 1, 2$:

$$||\nabla^j\mathfrak{R}|| \leq C\mu\varrho^{2-j},\ \varrho = \mathscr{W},\ \psi = \frac{1}{2}\log(g),\ g = \det((g_{ij}))$$

Integration by parts after multiplication by $\zeta^2 F_n$ and incorporation of the preceding estimates along with Young's inequality leads to:

$$\int_{\mathscr{W}} \zeta^2 |\nabla F_n|^2 \leq C\mu\varrho^2 \int_{\mathscr{W}} \zeta^2 |\nabla F_{n-1}|^2 + C_2 \int_{\mathscr{W}} \left(|\nabla\zeta|^2 + \zeta^2\right) F_n^2$$

and

$$\mathrm{supp}(|\nabla\zeta|) \subset (\mathscr{W}) = \{x \in \mathscr{W} / d(x, \partial\mathscr{W}) < \epsilon\}$$

and

$$|\nabla^j \zeta| \leq \frac{C_j}{\epsilon^j}$$

We select $C\mu\rho^2 = 1$ then

$$\int_{\mathscr{W}} \zeta^2 |\nabla F_n|^2 \leq C \int_{\mathscr{W}} |\nabla(\zeta F_0)|^2$$

Similarly we have the inequalities:

$$\int_{\mathscr{W}} \zeta^2 |\nabla^2 F_n|^2 \leq C\rho^2 \left(\rho^2 \int_{\mathscr{W}} \zeta^2 |\nabla^2 F_{n-1}|^2 + \int_{\mathscr{W}} \left(|\zeta| + |\nabla\zeta|^2\right) |\nabla F_{n-1}|^2 \right)$$

and

$$\int_{\mathscr{W}} \zeta^2 |\nabla^3 F_n|^2 \leq C^2\rho^2 \left(\rho^2 \int_{\mathscr{W}} \zeta^2 |\nabla^3 F_{n_1}|^2 + \int_{\mathscr{W}} \zeta^2 |\nabla^2 F_{n-1}|^2 + \right.$$

$$\left. + \rho^2 \int_{\mathscr{W}} \left(|\zeta| + |\nabla\zeta|^2\right) |\nabla^2 F_{n-1}|^2 \right.$$

Therefore we have that after iteration:

$$\int_{\mathscr{W}} \zeta^2 |\nabla^2 F_n|^2 \leq C \int_{\mathscr{W}} \zeta^2 |\nabla^2 F_0|^2$$

and as well as

$$\int_{\mathscr{W}} \zeta^2 |\nabla^3 F_n|^2 \leq C \int_{\mathscr{W}} \zeta^2 |\nabla^3 F_0|^2$$

The Nash-Moser iteration that we describe in the sequel allows us to bound the sequence in $C_0^2(\mathscr{W})$. Rellich lemma allows us to extract a sequence that converges in $H^1(\mathscr{W})$ and we can bound in..

$L_\infty - L^p$ estimate

g, χ smooth functions, $\widehat{h}$ polynomial weight function of degree m:

$$\mathscr{W}_j = \{x \in \mathscr{W} / \quad \theta(1-\theta^j)\frac{\eta}{2} \le |\widehat{h}(x)| \le (1-\theta+\theta^j)\eta\}$$

and $\mathrm{supp}\chi_j \subset \mathscr{W}_j$

$$\chi_j(x) = \ell\left(\frac{\widehat{h}(x)}{(1-\theta+\theta^j)\eta}\right)\ell\left(\frac{\theta(1-\theta^j)\eta}{\widehat{h}(x)}\right)$$

Smooth g satisfies the inequality, for positive constants $\gamma > 1, e = 2, 4$ and any smooth cut-off χ:

$$\int_{\mathscr{W}} \chi^2 |\nabla g|^2 \le \gamma \int_{\mathscr{W}} \chi^2 |g|^e$$

and

$$\mathcal{D}(\eta,\gamma) = (\eta^s\gamma^q)^3, \qquad s = \ell p + \frac{a+1}{3a}, \qquad q = \frac{p(t+1)}{2t} + \frac{t}{3}$$

Morrey type inequality

$\epsilon < 1,\ 0 < \gamma < 1$ or $\gamma < 0,\ p < 2$:

$$u_\epsilon = \sqrt{u^2 + \varepsilon^2}, \qquad \psi_\varepsilon = \log u_\varepsilon, \qquad w = u_\epsilon^\gamma$$

and $\zeta, \operatorname{supp}(\zeta) \subset \mathscr{W}$:

$$\int_{\mathscr{W}} \zeta^2 |\nabla u_\epsilon|^2 \leq C_0 \int_{\mathscr{W}} \zeta^2 u_\epsilon^2 \tag{4}$$

For $q = \frac{2}{\gamma}$:

$$\int_{\mathscr{W}} |\nabla w|^p \zeta^p \leq C_1 \int_{\mathscr{W}} |\nabla \zeta|^2 w^q \tag{5a}$$

$$\int_{\mathscr{W}} |\nabla \psi_\varepsilon|^2 \zeta^2 \leq C_2 \int_{\mathscr{W}} |\nabla \zeta|^2 + \zeta^2 \tag{5b}$$

(**??**) follows after selection for ζ as $u_\epsilon^{\gamma-1}$

$$|\nabla w| = \gamma w^{1+\frac{1}{\gamma}} |\nabla u_\epsilon|$$

and

$$\int \zeta^2 |\nabla w|^2 \leq \frac{C_0}{\cdot} \int \zeta^2 w^2$$

The inequality (**??**) requires the additional assumption for $\tau > 0$:

$$\int_{\mathscr{W}} \zeta^2 |\nabla^2 u|^2 \leq \tau \int_{\mathscr{W}} \zeta^2 u^2$$

We start selecting values $u_1, \ldots, u_m > 0$ and assume that

$$u = u_j + h_j$$

Assumptions:

- We set

$$(u_j + h_j)^2 \geq (1-\epsilon)\left(u_j^2 - \frac{h_j}{\epsilon^2}\right) \geq \theta^2 h_j^2$$

- We select

$$\frac{\theta^2\epsilon^2 + 1}{1-\epsilon^2} = c\epsilon^2$$

- We choose

$$c = 4 + \mu$$

- Finally $\epsilon = \frac{1}{2} + \mu$

Harmonic approximation h_j in suitable bricks selected so that we use the initial form of $\widehat{h}_j$. Hence we have that for $\psi = \log(u), \tilde{\psi} = \log(h)$

$$\int_{\mathscr{B}} |\nabla\psi|^2\zeta^2 \leq c \int_{\mathscr{B}} |\nabla\tilde{\psi}|^2\zeta^2$$

$\widehat{h}$ harmonic approximation of h in $\mathscr{W}$ and set:

$$h = \widehat{h} + \kappa$$

The standard harmonic approximation method estimates combined with partial integration leads us to

$$\int_{\mathscr{W}} \zeta^2|\nabla^2\kappa| \leq \mathcal{D}(\eta,\tau) \int_{\mathscr{W}} \zeta^2 u^2$$

We could get

$$\sup_{\mathscr{W}_0} |\nabla\kappa| \leq \mathcal{D}(\eta,\tau)\epsilon \left(\int_{\mathscr{W}} \zeta^2 u^2\right)^{1/2}$$

We compute for $\epsilon < 1$:

$$u_\epsilon^2 = \widehat{u}^2 + 2\kappa\widehat{u} + \kappa^2 + \epsilon^2 > c_\epsilon \widehat{u}^2$$

$$u_\epsilon = \sqrt{u^2 + \varepsilon^2}, \qquad \psi_\varepsilon = \log u_\varepsilon, \qquad w = u_\epsilon^\gamma$$

and for $\zeta, \operatorname{supp}(\zeta) \subset \mathscr{W}$:

$$\int_{\mathscr{W}} \zeta^2 |\nabla u_\epsilon|^2 \leq C_0 \int_{\mathscr{W}} \zeta^2 u_\epsilon^2 \tag{6}$$

Then for $q = \frac{2}{\gamma}$:

$$\int_{\mathscr{W}} |\nabla w|^p \zeta^p \leq C_1 \int_{\mathscr{W}} |\nabla \zeta|^2 w^q \tag{7a}$$

$$\int_{\mathscr{W}} |\nabla \psi_\varepsilon|^2 \zeta^2 \leq C_2 \int_{\mathscr{W}} |\nabla \zeta|^2 + \zeta^2 \tag{7b}$$

Hardy inequalities

Let P homogeneous polynomial of degree m and $\zeta \in C_0^\infty(\mathbf{R}^3 \setminus \{P = 0\})$ then

$$\int_{\mathscr{W}} \left| \frac{\nabla P}{P} \right|^2 \zeta^2 \le C_{1H} \int_{\mathscr{W}} |\nabla \zeta|^2$$

$$\int_{\mathscr{W}} P^{-\frac{2}{m}} \zeta^2 \le C_{2H} \int_{\mathscr{W}} |\nabla \zeta|^2$$