Homogeneous Inflationary String Cosmological Model in Brans-Dicke Theory

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Abstract

Homogeneous and isotropic inflationary string cosmological model is investigated in Brans-Dicke scalar tensor theory of gravitation. A special model is also obtained. For determinate solution, we have assumed that the trace of energy momentum tensor of string cloud vanishes and expansion θ is proportional to Eigen value σ_1^1 of shear tensor σ_i^j .The physical and kinematical parameters of the models are also discussed.

1. Introduction

In recent years, the researchers have shown their keen interest in the alternative theories of gravitation. One of the most popular theories among them is a scalar tensor theory of gravitation, proposed by Brans and Dicke [1]. This theory involves a scalar field ϕ in addition to the familiar general relativistic tensor. They postulated that ϕ behaves as the reciprocal of gravitational constant G, where ϕ is accepted to satisfy a scalar wave equation whose source is all the matter in the universe. Einstein pointed out that Mach's principle is not sustained by general relativity. Now the study of cosmological models of Brans-Dicke theory, which develops Mach's principle in a relativistic framework by assuming interaction of inertial masses of fundamental particles with some cosmic scalar field ϕ coupled with large-scale distribution of matter in motion, has gained momentum. The latest inflationary models[2] has been studied by Mathiazhagan and Johri, extended inflation by La and Steinhardt [3], Steinhardt and Accetta[4], hyper extended inflation and extended chaotic inflation by Linde[5] are based on Brans-Dicke theory and general scalar tensor theories.

Several authors have studied Brans-Dicke cosmological models in four dimensions. Singh and Rai [6] investigated Brans-Dicke cosmological models with perfect fluid as a source in detail. The string theory has a great importance to describe an event at the early stage of evolution of the universe in a lucid way. According to Kibble[7] and Zel'dovich[8], the cosmic strings arise during the phase transitions as the universe passes through its critical temperature after the big-bang explosion and give rise to perturbation leading to the formation of galaxies .The cosmic strings have stress energy and couple to the gravitational field and gravitational field is related to scalar field ϕ. Therefore, it is interesting to study scalar field effect, which arises from string by using Brans-Dicke field equations.

The general treatment of strings was initiated by Letelier[9,10] and Stachel[11]. In recent years various authors viz. Vilenkin[12], Krori et al.[13,14], Chakraborty and Chakraborty[15], Tikekar and Patel[16,17], Bali and Upadhyaya[18], Bali and Singh[19], Bali and Anjali[20], Pradhan et al.[21],Rathore et al. [22] have investigated string cosmological models in general relativity whereas Sen[23], Mahanta and Mukherjee[24], Bhattacharjee and Baruah[25], Bali et al.[26] have studied string cosmological models in alternative theories of gravitation in four dimensions. Rahaman et al. [27], Mohanty and Mahanta [28], Rathore and Mandawat [29,30] investigated string cosmological models in alternative theories of gravitation in higher dimensions.

The present investigation is concerned with Homogeneous string cosmological model having inflationary character in Brans-Dicke scalar tensor theory of gravitation, when the trace of energy momentum tensor of string cloud vanishes i.e. $\rho+\lambda=0$, Here ρ is the rest energy density and λ is the tension density. To get a determinate solution, we have assumed a physically plausible condition that the expansion (θ) is proportional to Eigen value (σ_1^1) of shear tensor (σ_i^j), which is physically plausible condition, where

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}$$

$$\sigma_1^1 = \frac{1}{3}\left(\frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C}\right)$$

Thus $\theta \propto \sigma_1^1$ leads to

$$\frac{A_4}{A} = \frac{\ell+3}{2\ell-3}\left(\frac{B_4}{B} + \frac{C_4}{C}\right)$$

which again leads to

$$A = \ell_1 (BC)^n$$

where ℓ is proportional constant and ℓ_1 is constant of integration and $n = \frac{\ell+3}{2\ell-3}$. Some physical and kinematical properties of the model are also discussed.

2. The Metric and Field Equations

We consider the spatially homogeneous and anisotropic Bianchi type-I metric in the form.

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \qquad (1)$$

where A, B, C are functions of cosmic time 't' alone.

The Brans-Dicke scalar tensor field equations are given by

$$G_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}\,\phi_{,k}\,\phi^{,k}) - \phi^{-1}(\phi_{i;j} - g_{ij}\,\Box\phi) \qquad (2)$$

And $\Box \phi = \phi^{;k}_{;k} = 8\pi\phi^{-1}T(3+2\omega)^{-1}$

(3)

where $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$ is the Einstein tensor, T_{ij} is the stress Energy of matter, ϕ is the scalar field and ω is the dimensionless coupling constant and comma and semicolon denote partial and covariant differentiation respectively.

The equation of motion

$$T^{ij}_{;j} = 0 \tag{4}$$

are consequence of the field equation.

The energy momentum tensor for string dust source is given by

$$T^i_j = \rho u^i u_j - \lambda x^i x_j \tag{5}$$

where u^i is the four velocity of the string cloud, x^i is the normal space like four vector which represents the string direction i.e. direction of anisotropy and ρ is the rest energy density of the cloud of strings with particles attached to them and λ is tension density of the cloud of strings. The string source is along the x-axis, which is the axis of symmetry. Ortho normalization of u^i and x^i is given as

$$u^i u_i = x^i x_i = -1 \text{ and } u^i x_i = 0 \tag{6}$$

In the co-moving coordinate system, we have from (5)

$$T^1_1 = -\lambda, \quad T^2_2 = T^3_3 = 0, \quad T^4_4 = -\rho$$
$$T^i_j = 0, \quad \text{for } i \neq j \text{ and } T = -(\rho + \lambda) \tag{7}$$

We also consider

$$\rho = \rho_p + \lambda$$

where ρ_p is the rest energy density of the particles. The energy densities for coupled system ρ and ρ_p are restricted to be positive and the tension density λ may be positive or negative. Here the quantities ρ, λ and the scalar field are functions of cosmic time only.

The Brans-Dicke field equations (2), (3) and (4) for the metric (1) lead to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = 8\pi\phi^{-1}\lambda - \frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) \tag{8}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} = -\frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{C_4}{C} + \frac{A_4}{A}\right) \tag{9}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -\frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{A_4}{B} + \frac{B_4}{B}\right) \tag{10}$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} = 8\pi\phi^{-1}\rho + \frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^4 + \frac{\phi_4}{\phi}\left(\frac{A_4}{B} + \frac{B_4}{B} + \frac{C_4}{C}\right) \tag{11}$$

$$\Box\phi = \phi_{44} + \phi_4\left[\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right] = (\rho + \lambda)\frac{8\pi\phi^{-1}}{(3 + 2\omega)} \tag{12}$$

$$\rho_4 + (\rho - \lambda)\frac{A_4}{A} + \rho\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = 0 \tag{13}$$

where the sub indices '4' denotes differentiation with respect to t.

3. Solution of Field Equations

From equation (9) and (10), we have

$$\frac{BC_{44} - CB_{44}}{BC_4 - CB_4} + \frac{A_4}{A} = \frac{\phi_4}{\phi} \tag{14}$$

Using condition

$$A = \ell_1(BC)^n \tag{15}$$

in (14), we have

$$\frac{BC_{44} - CB_{44}}{BC_4 - CB_4} + n\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{\phi_4}{\phi}$$

On integration, which leads to

$$B^2\left(\frac{C}{B}\right)_4 = \frac{k_1\phi}{(BC)^n} \tag{16}$$

Let BC = μ and C/B = ν

Therefore, we have $B^2 = \frac{\mu}{\nu}$, $C^2 = \mu\nu$

Now, from equation (16), we have

$$\frac{\nu_4}{\nu} = \frac{K_1\phi}{\mu^{n+1}} \tag{17}$$

where K_1 is the constant of integration.

From equation (12), with the help of (15)

$$\phi_4 = \frac{K_2}{\mu^{n+1}} \tag{18}$$

where K_2 is the constant of integration.

From the equations (8) and (11), together with the condition $\rho + \lambda = 0$ and equation (18) we have

$$\frac{(BC_4 + CB_4)_4}{BC_4 + CB_4} + n\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{\phi_4}{\phi}$$

On integration, it leads to

$$\phi = \frac{\mu_4\mu^n}{K_3} \tag{19}$$

where K_3 is the constant of integration.

Equation (18) and (19) leads to

$$\mu\mu_{44} + n\mu_4^2 = K\mu^{-2n} \tag{20}$$

where $K = K_2 K_3$

Which leads to

$$ff' + \frac{n}{\mu} f^2 = K\mu^{-2n-1} \tag{21}$$

where $\mu_4 = f(\mu)$ and $f' = \frac{df}{d\mu}$

From equation (21), we have

$$f^2 = \frac{1}{\mu^{2n}}(2K \log \mu + K_4) \tag{22}$$

where K_4 is the constant of integration.

$$\mu_4 = \frac{1}{\mu^n}\sqrt{(2K \log \mu + K_4)} \tag{23}$$

From equation (17), we get

$$v = K_5(\mu)^{\frac{K_1}{K_2}} \tag{24}$$

where K_5 is the constant of integration.

The metric (1) reduces to

$$ds^2 = -dt^2 + \mu^{2n}dx^2 + \frac{\mu}{v}dy^2 + \mu v dz^2 \tag{25}$$

Using the transformation

$$\mu = T,\ x = X,\left(\frac{1}{\sqrt{K_5}}\right)y = Y,\ \sqrt{K_5}z = Z$$

metric (25) reduces to

$$ds^2 = -\frac{dT^2}{\frac{1}{T^{2n}}(2K \log T + K_4)} + T^{2n}dX^2 + T^{1-\frac{k_1}{k_3}}dY^2 + T^{1+\frac{k_1}{k_3}}dZ^2 \tag{26}$$

4. Some Physical and Geometrical Properties

The energy density (ρ), String tension density (λ) and the scalar field (ϕ) for the model (26) are given by

$$\rho = \frac{K_6}{T^{2n+1}} \tag{27}$$

$$\lambda = -\frac{K_6}{T^{2n+1}} \tag{28}$$

$$\phi = \frac{1}{K_3}\sqrt{2K\log T + K_4} \tag{29}$$

The scalar of expansion (θ), the non-vanishing components of shear (σ_i^j), shear (σ) and spatial volume (V) and deceleration parameter (q) for model (26) are given by

$$\theta = \frac{(n+1)}{T^{n+1}}\sqrt{2K\log T + K_4} \tag{30}$$

$$\sigma_1^1 = \frac{(2n-1)}{3}\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \tag{31}$$

$$\sigma_2^2 = \left[\left(\frac{1-2n}{6}\right) - \frac{1}{2}\frac{K_1}{K_2}\right]\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \tag{32}$$

$$\sigma_3^3 = \left[\left(\frac{1-2n}{6}\right) + \frac{1}{2}\frac{K_1}{K_3}\right]\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \tag{33}$$

$$\sigma = \left[\frac{1}{12}(2n-1)^2 + \frac{1}{2}\frac{K_1^2}{K_3^2}\right]^{\frac{1}{2}}\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \tag{34}$$

$$V = T^{n+1} \tag{35}$$

$$q = -\left[\frac{3\{K - (2K\log T + K_4)\}}{(n+1)(2K\log T + K_4)}\right] - 1 \tag{36}$$

The model (26) starts with big-bang at T = 0. For (n + 1) > 0 the expansion in the model decreases as the time increases where as for (n+1) < 0 the expansion increases as time increases. The model has a point type singularity (MacCallum[31]) at T = 0 for n > 0 with the condition $\frac{K_1}{K_2} \in (-1,1)$ and cigar type singularity at T = 0 for n < 0 with $\frac{K_1}{K_2} \in (-1,1)$ or n > 0 with $\frac{K_1}{K_3} > 1$ and Pan cake singularity at T = 0 for $\frac{K_1}{K_2} \in (-\infty,-1) \cup (1,\infty)$.

The rest energy density ρ→ ∞ as T→ 0 but ρ→0 as T→∞ when $K_6 > 0$ and n > – ½. The string tension density λ → 0 as T →∞ but λ → –∞ as T → 0 for $K_6 > 0$ and n > – ½. Therefore rest energy density and string tension density of the model has initial singularity at T = 0. The rest energy density for the model (26) satisfies the reality condition ρ > 0 given by Ellis if $K_6 > 0$. The Brans-Dicke scalar field ϕ increases indefinitely as time T → ∞ and it blows up at its initial stage. The spatial volume V of the model given by (35) shows that the model follows power law inflation. The model isotropizes at all epoch for n = ½ and $K_1 = 0$, but remain anisotropic throughout the evolution when n ≠ ½ or $K_1 \neq 0$ or both.

The cosmological model has singularities of different kinds in different conditions at initial epoch, where as rest Energy density, tension density and Brans-Dicke scalar field are not free from initial singularities. The sign of deceleration parameter q is different in different time zones. Therefore, it is concluded that the universe will witness an epoch when it transits from the phase of acceleration to deceleration of vice-versa. Hence, the present cosmological model represents an inflationary, shearing, non-rotating and anisotropic string cosmological model in Brans-Dicke scalar-tensor theory of gravitation and behaves in accordance with the present observational data. Therefore, this cosmological model is a realistic model.

5. Special Model

Here we are considering the case for which n = -1. This case leads to $\ell = 0$, which restricts us to consider the assumption that expansion is proportional to the shear component σ_1^1. Therefore we have solved the field equation by taking n = -1 in a different manners.

Putting n = -1 in the equation (23), we have

$$\mu_4 = \mu\sqrt{2K\log\mu + K_4}$$

On integration equation (37) reduces to

$$\mu = \exp\left[\frac{K}{2}(t+K_7)^2 - \frac{K_4}{2K}\right]$$

Where K_7 is the constant of integration.

and with the help of (24), we have

$$A = \exp\left[-\frac{K}{2}(t+K_1)^2 + \frac{K_4}{2K}\right]$$

$$B = \frac{1}{(K_5)^{\frac{1}{2}}}\exp\left[\frac{1}{2}\left(1-\frac{K_1}{K_3}\right)\left\{\frac{K}{2}(t+K_7)^2 - \frac{K_4}{2K}\right\}\right]$$

$$C = (K_5)^{\frac{1}{2}}\exp\left[\frac{1}{2}\left(1+\frac{K_1}{K_3}\right)\left\{\frac{K}{2}(t+K_1)^2 - \frac{K_4}{2K}\right\}\right]$$

and using suitable transformation of coordinates, the metric (1) reduces to

$$ds = -dT^2 + e^{-KT^2}dX^2 + e^{\left\{\frac{K}{2}\left(1-\frac{K_1}{K3}\right)T^2\right\}}dY^2 + e^{\left\{\frac{K}{2}\left(1+\frac{K_1}{K_3}\right)T^2\right\}}dZ^2$$

This special model has no initial singularity and for small values of T, it reduces to a spherically symmetric cosmological model. It is neither expanding nor contracting model but the shear increases as T increases and it remains always anisotropic. Therefore, it is not a realistic model.

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