Global study of linear waves of Schwarzschild de Sitter spacetimes

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Introduction

- The primary impetus for the study of the linear wave equation on black hole spacetimes is that some aspects of the stability or instability of dynamical black holes are already foreshadowed in the global behaviour of linear waves.
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Introduction

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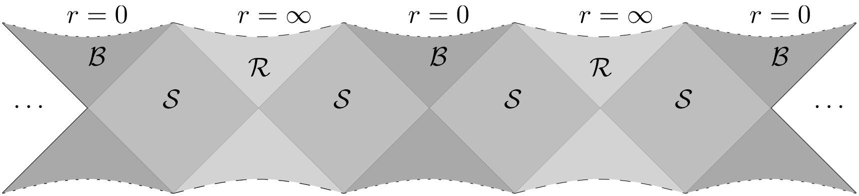
  (cf. talks of Dafermos, Aretakis, and Civin)
- An additional stability mechanism is expected to be present for *expanding* spacetimes, namely solutions to the field equations with *positive* cosmological constant.
  `Friedrich` ('86), . . . ,`Ringström` ('09),
  `Rodnianski-Speck` ('12)

Schwarzschild de Sitter spacetimes

- The Schwarzschild de Sitter spacetimes $(\mathcal{M}_{(\Lambda,m)}, g)$ are a family of *spherically symmetric* black hole spacetimes parametrized by the cosmological constant $\Lambda > 0$, and the mass $0 < m < \frac{1}{3\sqrt{\Lambda}}$.
- We may think of this spacetime as an infinite chain of black hole interiors, *static* black hole exterior regions, and *expanding* or *cosmological* regions separated by event horizons and cosmological horizons respectively:



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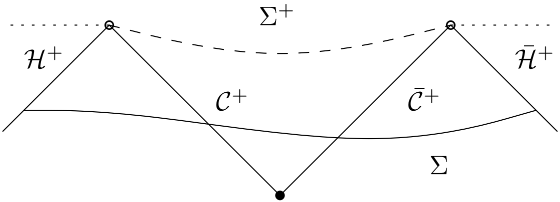
- My work focuses on the expanding region $\mathcal{R}$, and uses a result of `Dafermos-Rodnianski` ('07) for the static region, to obtain a *global* result.

Linear waves on Schwarzschild de Sitter

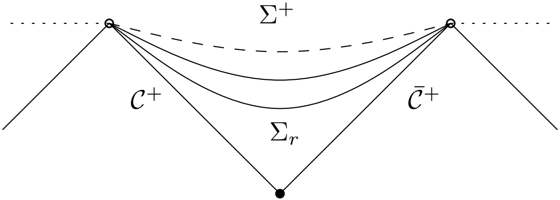
- The expanding region $\mathcal{R}$ is bounded in the past by cosmological horizons $\mathcal{C}^+$, $\bar{\mathcal{C}}^+$ and to the future by a spacelike hypersurface Σ^+.
- The Cauchy problem for the wave equation that I study,

$$\Box_g \psi = 0 \,,$$

is posed in the past of Σ^+, with initial data prescribed on a spacelike hypersurface Σ as follows:



Linear waves on Schwarzschild de Sitter



- The expanding region $\mathcal{R}$ is foliated by the level sets Σ_r of the area radius function r. Each leaf Σ_r is topologically a cylinder, and embedded as a spacelike hypersurface. We can think of the future boundary Σ^+ to be endowed with the standard metric of the cylinder $\mathbb{R} \times \mathbb{S}^2$.
- In my recent work I exhibit an energy which is *monotone* with respect to this foliation for solutions to the linear wave equation.

Global boundedness result for linear waves

Theorem (VS '12)

Let Σ be a Cauchy hypersurface with normal n for the Cauchy problem discussed before. Then all solutions ψ to the wave equation with initial finite energy on Σ,

$$D[\psi] \doteq \int_\Sigma T[\psi](n,n)\,\mathrm{d}\mu_{\overline{g}} < \infty\,,$$

are globally bounded on $\overline{\mathcal{R}}$ and have a limit on Σ^+. Moreover, the limit *as a function on* $\mathbb{R}\times\mathbb{S}^2$ satisfies

$$\int_{\Sigma^+} \big|\mathring{\nabla}\,\psi\big|^2 \mathrm{d}\mu_{\mathring{g}} \leq C(\Lambda, m, \Sigma)\,D[\psi]\,,$$

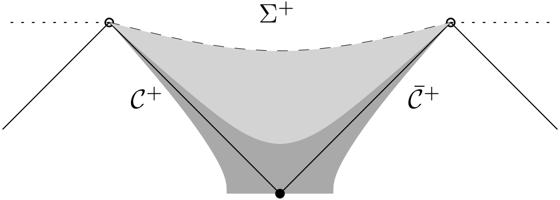
where C is a constant that only depends on Λ, m, and Σ.

Global boundedness result for linear waves

Remarks:

- Since we establish the boundedness of a limiting *rescaled* quantity, my result can be read as an implicit version of a decay statement for the "natural" energy in this problem.
- Moreover the result is in agreement with our expectation for the nonlinear stability problem. Although we expect to recover the same global *causal* geometry, the dynamical development of a perturbation of Schwarzschild de Sitter initial data is not expected to settle down to the *exact* geometry of the expanding region a Schwarzschild de Sitter solution. This is captured by a nonvanishing bounded quantity on the future boundary.

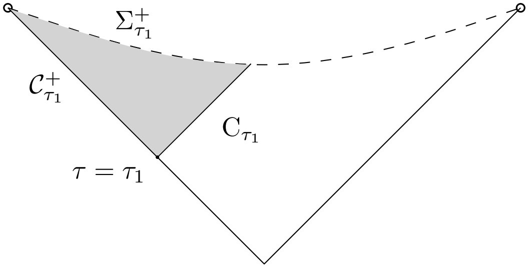
Global boundedness and redshift



Qualitatively the result is a consequence of the presence of a redshift effect in the expanding region:

- the *global* redshift effect (region in light grey) due to the expansion of the spacetime,
- and the *local* redshift effect near the cosmological horizon (in dark grey) due to the positivity of the surface gravity.

Decay of linear waves along the future boundary



Notation:

- Let τ be a function on $\mathcal{C}^+$ which is constant on the spheres of symmetry such that $\tau = \tau_0 > 0$ on a chosen sphere $S \subset \mathcal{C}^+$ to the future of $\mathcal{C}^+ \cap \bar{\mathcal{C}}^+$ and

$$T \cdot \tau \Big|_{\mathcal{C}^+} = 1 \, .$$

Decay of linear waves along the future boundary

Corollary

Assume ψ is a solution to the wave equation which decays polynomially along $\mathcal{C}^+$ in the sense that

$$\text{“ incoming energy flux through } \mathcal{C}^+_\tau \text{ ”} \leq \frac{C_k}{\tau^k} \qquad (\tau > \tau_0)\,. \tag{A}$$

Then there exists a constant $C_k \leq C < \infty$, and $\tau_0 < \tau_1 < \infty$ such that also

$$\int_{\Sigma^+_\tau} \left|\overset{\circ}{\nabla}\psi\right|^2 \mathrm{d}\mu_{\overset{\circ}{g}} \leq \frac{C}{\tau^k} \qquad (\tau > \tau_1)\,. \tag{D}$$

Moreover, if the decay in (A) is assumed to be exponential, then also the decay along the future boundary in (D) is exponential.

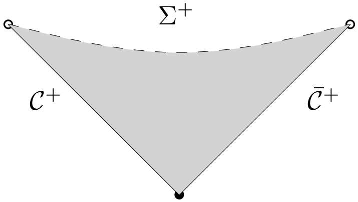
Decay of linear waves along the future boundary

Remarks:

- The assumptions of the Corollary are known to be satisfied in the context of the global Cauchy problem as discussed earlier *under additional conditions on the higher order energies* of the initial data.
- The explicit condition for polynomial decay along the horizon is given in `Dafermos-Rodnianski` ('07).
- An exponential decay result under suitable assumptions on the initial data was obtained by `Dyatlov` ('10).

Open problem

The linear study is ultimately motivated by the nonlinear stability problem for the expanding region.



It is possible to study the characteristic initial value problem for the Einstein equations with initial data given by small perturbations of the geometry of the cosmological horizons of Schwarzschild de Sitter.