Gravito-electromagnetic resonances

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NEB XV: Hania, June 20, 2012

Main Objective/Question

Consider the interaction between the Weyl and the Maxwell fields

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Consider the interaction between the Weyl and the Maxwell fields

Investigate the possibility of resonances?

Gravitational waves

1+3 covariant description

- Relative to an observer with 4-velocity u_a ($u_a u^a = -1$).
- By the Weyl part of the curvature: the electric (E_{ab}) and magnetic (H_{ab}) Weyl tensors.
- With $E_{ab} = E_{\langle ab \rangle}$, $H_{ab} = H_{\langle ab \rangle}$ and $\mathrm{D}^b E_{ab} = 0 = \mathrm{D}^b H_{ab}$.

The role of the shear

- In highly symmetric spacetimes (e.g. Minkowski, FRW) E_{ab} , $H_{ab} \to \sigma_{ab}$.
- There, gravitational waves are monitored by the shear, with $\sigma_{ab} = \sigma_{\langle ab \rangle}$ and $\mathrm{D}^b \sigma_{ab} = 0$.

Isolating gravitational waves

- The conditions $\mathrm{D}^b E_{ab} = 0 = \mathrm{D}^b H_{ab} = \mathrm{D}^b \sigma_{ab}$ must be satisfied at all times.
- On Minkowski and FRW backgrounds, we set $A_a = 0 = \omega_a = \mathrm{D}_a \rho = \mathrm{D}_a \Theta$ (to 1st order).

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Source-free electromagnetic fields

1+3 covariant description

- Relative to an observer with 4-velocity u_a ($u_a u^a = -1$).
- By the electric (E_a) and magnetic (B_a) fields.

Faraday tensor

$$F_{ab} = 2E_{[a}u_{b]} + \varepsilon_{abc}B^c\,.$$

Maxwell's equations

- Ampere's law: $\dot{E}_{\langle a\rangle} = -\frac{2}{3}\Theta E_a + (\sigma_{ab} + \omega_{ab})E^b + \varepsilon_{abc}A^b B^c + \mathrm{curl}B_a\,.$
- Faraday's law: $\dot{B}_{\langle a\rangle} = -\frac{2}{3}\Theta B_a + (\sigma_{ab} + \omega_{ab})B^b - \varepsilon_{abc}A^b E^c - \mathrm{curl}E_a\,.$
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Gravitational waves on Minkowski space

Shear wave equation

At the linear level

$$\ddot{\sigma}_{ab} - \mathrm{D}^2\sigma_{ab} = 0\,.$$

Harmonic splitting

Set

$$\sigma_{ab} = \sum_k \sigma_{(k)}\mathcal{Q}^{(k)}_{ab}\,,$$

with

$$\mathrm{D}_a\sigma_{(k)} = 0 = \dot{\mathcal{Q}}^{(n)}_{ab} \qquad \text{and} \qquad \mathrm{D}^2\mathcal{Q}^{(k)}_{ab} = -k^2\mathcal{Q}^{(k)}_{ab}\,.$$

Solution

Then,

$$\ddot{\sigma}_{(k)} + k^2\sigma_{(k)} = 0$$

and

$$\sigma_{(k)} = \mathcal{W}\sin(kt+\varphi)\,.$$

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Set

$$E_a = \sum_n E_{(n)} \mathcal{Q}_a^{(1)(n)} \qquad and \qquad B_a = \sum_n B_{(n)} \mathcal{Q}_a^{(2)(n)}\,,$$

with

$$\mathrm{D}_a E_{(n)} = 0 = \mathrm{D}_a B_{(n)} = \dot{\mathcal{Q}}_a^{(i)(n)} \qquad and \qquad \mathrm{D}^2 \mathcal{Q}_a^{(i)(n)} = -n^2 \mathcal{Q}_a^{(i)(n)}\,, \quad (i = 1, 2)\,.$$

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$$\ddot{E}_{(n)} + n^2 E_{(n)} = 0 \qquad and \qquad \ddot{B}_{(n)} + n^2 B_{(n)} = 0\,,$$

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Gravito-electromagnetic interaction

Gravitationally driven EM waves

To second order

$$\ddot{E}_a - \mathrm{D}^2 E_a = \sigma_{ab}\dot{\tilde{E}}^b + \varepsilon_{abc}\tilde{B}_d \mathrm{D}^b \sigma^{cd} - 2\varepsilon_{abc}\sigma^b{}_d \mathrm{D}^{\langle c}\tilde{B}^{d\rangle} - \mathcal{R}_{ab}\tilde{E}^b - E_{ab}\tilde{E}^b + H_{ab}\tilde{B}^b$$

and

$$\ddot{B}_a - \mathrm{D}^2 B_a = \sigma_{ab}\dot{\tilde{B}}^b - \varepsilon_{abc}\tilde{E}_d \mathrm{D}^b \sigma^{cd} + 2\varepsilon_{abc}\sigma^b{}_d \mathrm{D}^{\langle c}\tilde{E}^{d\rangle} - \mathcal{R}_{ab}\tilde{B}^b - E_{ab}\tilde{B}^b - H_{ab}\tilde{E}^b\,,$$

where $\tilde{E}_a$, $\tilde{B}_a$ and E_a, B_a are the original and the gravitationally driven EM fields.

Auxiliary relations

To first order

$$\mathcal{R}_{ab} = E_{ab}\,, \qquad E_{ab} = -\dot{\sigma}_{ab}$$

and

$$H_{ab} = \mathrm{curl}\sigma_{ab}\,.$$

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Simplify

Ignoring “backreaction” terms

$$\ddot{E}_a - \mathrm{D}^2 E_a = \sigma_{ab}\dot{\tilde{E}}^b + 2\dot{\sigma}_{ab}\tilde{E}^b$$

and

$$\ddot{B}_a - \mathrm{D}^2 B_a = \sigma_{ab}\dot{\tilde{B}}^b + 2\dot{\sigma}_{ab}\tilde{B}^b\,.$$

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The gravitationally driven EM wave

Monochromatic waves

$$\ddot{E}_{(\ell)} + \ell^2 E_{(\ell)} = \sigma_{(k)} \dot{\tilde{E}}^{(n)} + 2\dot{\sigma}_{(k)} \tilde{E}^{(n)}\,,$$

$$\ddot{B}_{(\ell)} + \ell^2 B_{(\ell)} = \sigma_{(k)} \dot{\tilde{B}}^{(n)} + 2\dot{\sigma}_{(k)} \tilde{B}^{(n)}\,,$$

where

$$\tilde{E}_{(n)}, \tilde{B}_{(n)} = \mathcal{M} \sin(nt + \vartheta)\,, \qquad \sigma_{(k)} = \mathcal{W} \sin(kt + \varphi)\,,$$

and

$$\ell^2 = k^2 + n^2 + 2kn \cos\phi\,.$$

Forced oscillations

For ϑ, $\varphi = 0$,

$$\ddot{E}_{(\ell)} + \ell^2 E_{(\ell)} = \mathcal{C}_+ \sin[(k+n)t] + \mathcal{C}_- \sin[(k-n)t]\,,$$

$$\ddot{B}_{(\ell)} + \ell^2 B_{(\ell)} = \mathcal{C}_+ \sin[(k+n)t] + \mathcal{C}_- \sin[(k-n)t]\,,$$

with $\mathcal{C}_\pm = \mathcal{M}\mathcal{W}(n \pm 2k)/2$.

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Resonant solutions

The $k = n$ and $\varphi = 0 = \vartheta$ case

$$\ddot{E}_{(\ell)} + \ell^2 E_{(\ell)} = \mathcal{F} \sin(mt)\,,$$

where $\mathcal{F} = 3k\mathcal{M}\mathcal{W}/2$ and $m = 2k$. Then,

$$E_{(\ell)} = \mathcal{C}_1 \sin(\ell t) + \mathcal{C}_2 \cos(\ell t) + \frac{3k\mathcal{M}\mathcal{W}}{2(\ell^2 - m^2)} \sin(mt)\,.$$

Resonance as $\ell \to m$.

The $k \neq n$ and $\varphi \neq \vartheta \neq 0$ case

$$\begin{aligned} E_{(\ell)} &= \mathcal{C}_1 \sin(\ell t) + \mathcal{C}_2 \cos(\ell t) \\ &\quad + \frac{\mathcal{F}_1}{\ell^2 - m_1^2} \sin(m_1 t + \omega_1) + \frac{\mathcal{F}_2}{\ell^2 - m_2^2} \sin(m_2 t + \omega_2)\,, \end{aligned}$$

where $\mathcal{F}_{1,2} = \mathcal{M}\mathcal{W}(n \pm 2k)/2$, $m_{1,2} = k \pm n$ and $\omega_{1,2} = \varphi \pm \vartheta$.

Resonances as $\ell \to m_{1,2}$.

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Resonances as $\ell \to m_{1,2}$.

The Weyl-Maxwell coupling in cosmology

The gravitomagnetic system

To linear order

$$\ddot{\sigma}_{(k)} + 5H\dot{\sigma}_{(k)} + \left[\frac{3}{2}(1-3w)H^2 + \left(\frac{k}{a}\right)^2\right]\sigma_{(k)} = 0 \qquad \textit{and} \qquad \tilde{B}_{(n)},\, \tilde{E}_{(n)} \propto a^{-2}\,.$$

To second order,

$$\ddot{B}_{(\ell)} + 5H\dot{B}_{(\ell)} + \left[3(1-w)H^2 + \left(\frac{\ell}{a}\right)^2\right]B_{(\ell)} = 2(\dot{\sigma}_{(k)} + 2H\sigma_{(k)})\tilde{B}_{(n)}\,.$$

The gravitationally induced B-field

During the radiation era,

$$B_{(\ell)} = \frac{1}{a^2}\left[C_{\pm}Si\left(\frac{\ell \mp k}{a_0 H_0}\sqrt{\frac{t}{t_0}}\right)\sin(\theta_{\mp}) + C_{\pm}Ci\left(\frac{\ell \mp k}{a_0 H_0}\sqrt{\frac{t}{t_0}}\right)\cos(\theta_{\mp})\right]\,.$$

Resonance as $\ell \to k$ and $Ci[(\ell - k)\sqrt{t}/a_0 H_0 \sqrt{t_0}] \to -\infty$.

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Resonance as $\ell \to k$ and $Ci[(\ell - k)\sqrt{t}/a_0H_0\sqrt{t_0}] \to -\infty$.

Interpreting resonances

Forced oscillation

$$\ddot{x} + \ell^2 x = \mathcal{F} \sin(mt)\,,$$

where $\mathcal{F}$ is the driving "force".

Typical resonance

$$x = C_1 \sin(\ell t) + C_2 \cos(\ell t) + \frac{\mathcal{F}}{\ell^2 - m^2} \sin(mt)\,.$$

The amplitude diverges ($x \to \infty$) as $m \to \ell$.

Standard interpretation

When $x_0 = 0 = \dot{x}_0$

$$x = \frac{\mathcal{F}}{\ell} \lim_{m \to \ell} \left[\frac{\ell \sin(mt) - m \sin(\ell t)}{\ell^2 - m^2} \right] = \frac{\mathcal{F}}{2\ell} \left[\sin(\ell t) - \frac{t}{\ell} \cos(\ell t) \right]\,.$$

Linear growth in time.

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Typical resonance

$$x = \mathcal{C}_1 \sin(\ell t) + \mathcal{C}_2 \cos(\ell t) + \frac{\mathcal{F}}{\ell^2 - m^2}\sin(mt)\,.$$

The amplitude diverges ($x \to \infty$) as $m \to \ell$.

Standard interpretation

When $x_0 = 0 = \dot{x}_0$

$$x = \frac{\mathcal{F}}{\ell}\lim_{m\to\ell}\left[\frac{\ell\sin(mt) - m\sin(\ell t)}{\ell^2 - m^2}\right] = \frac{\mathcal{F}}{2\ell}\left[\sin(\ell t) - \frac{t}{\ell}\cos(\ell t)\right]\,.$$

Linear growth in time.

Minkowski case ($n = k$, $\vartheta = 0 = \varphi$)

Solution with a singularity

$$E = \mathcal{C}_1 \sin(\ell t) + \mathcal{C}_2 \cos(\ell t) + \frac{3k\tilde{E}_0\sigma_0}{2(\ell^2 - m^2)} \sin(mk)$$

Solution without singularity

Set $E_0 = \tilde{E}_0$ and $\dot{E}_0 = \dot{\tilde{E}}_0$. Then, as $m \to \ell$

$$E = \frac{1}{\ell}\left(\dot{\tilde{E}} + \frac{3\tilde{E}_0\sigma_0}{8}\right)\sin(\ell t) + \tilde{E}_0\left(1 + \frac{3\sigma_0}{8}\,t\right)\cos(\ell t)\,.$$

Appreciable EM amplification (?)

- When close to the GW source.
- With high-frequency GWs.

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Underlying question

Possibility of energy transfer from GWs to EM fields

Widespread presence of EM fields - no GWs (so far)

Electromagnetism at the expense of gravitational radiation?

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