Geometrization of Lie and Noether symmetries with applications in Cosmology

Geometrization of Lie and Noether symmetries

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Plan

- We consider system of ODEs of the form

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = F^i \tag{1}$$

where Γ^i_{jk} are general functions, a dot over a symbol indicates derivation with respect to the parameter "s" along the solution curves. F^i is vector C^∞ field.

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  where Γ^i_{jk} are general functions, a dot over a symbol indicates derivation with respect to the parameter "s" along the solution curves. F^i is vector C^∞ field.
- This type of equations contains the equations of motion of a dynamical system in a Riemannian space if Γ^i_{jk} are the connection coefficients of the metric .In this case "s" is an affine parameter along the trajectory

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- a. The determination of the conditions, which the components of the Lie symmetry vector must satisfy
- b. The solution of the system of these conditions.
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- b. Transfer the problem to Differential Geometry and use well known theorems to solve the system of conditions geometrically in a general $n-$dimensional Riemannian space.

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- Then the generators of Lie point symmetries of (1) will be related to the generators of collineations of the metric
- In this way the determination of the Lie point symmetries of (1) is transferred to the geometric problem of determining the generators of a specific type of symmetries of the metric. In a way, the Lie point symmetry problem of (1) has been "geometrized".

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  The Lie point symmetries of (1) are amongst the generators of the special projective group of the space.This result is common to all dynamical systems moving in the specific space.
- The rhs of (1) specifies the dynamical system (the force).
- Therefore for each specific dynamical system 'moving' in a Riemannian space the equations of motion admit Lie symmetries if certain conditions hold between the force F^i and the generators of the special projective algebra of the space.

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- a. Either "select" the Lie symmetry generators from the special projective algebra of the space for a given dynamical system or
- b. Select the forces F^i which admit Lie symmetry generators from the special projective algebra of the space.
- Both types of answers are of interest and in the following we present some applications.

Noether symmetries

- If the system of equations (1) admits a Lagrangian L then the Lie symmetries which satisfy the additional condition

$$X^{[1]}L + L\dot{\xi} = \dot{f} \quad \text{where } f = f\left(t, x^i, \dot{x}^i\right) \tag{2}$$

are called Noether symmetries. For each Noether point symmetry the quantity

$$I = \xi\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) - \dot{x}_i \eta^i + f \tag{3}$$

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- **Theorem 2**

The Noether point symmetries are elements of the homothetic algebra of the space (which is a subalgebra of the special projective algebra). Furthermore the first integrals of a Noether point symmetry are linear in the velocities.

The projective algebra of a Riemannian space

Collineation	**A**	**B**
Killing vector (KV)	g_{ij}	0
Homothetic vector (HV)	g_{ij}	$\psi g_{ij},\ \psi_{,i} = 0$
Conformal Killing vector (CKV)	g_{ij}	$\psi g_{ij}, \psi_{,i} \neq 0$
Affine Collineation (AC)	Γ^i_{jk}	0
Projective collineation (PC)	Γ^i_{jk}	$2\phi_{(,j}\delta^i_{k)},\ \phi_{,i} \neq 0$
Special Projective collineation (SPC)	Γ^i_{jk}	$2\phi_{(,j}\delta^i_{k)},\ \phi_{,i} \neq 0$ and$\phi_{,jk} = 0$

The projective algebra of the Euclidian space

Collineation	Gradient	Non-gradient
Killing vectors (KV)	$\mathbf{S}_I = \delta^i_I \partial_i$	$\mathbf{X}_{IJ} = \delta^j_{[I} \delta^i_{j]} x_j \partial_i$
Homothetic vector (HV)	$\mathbf{H} = x^i \partial_i$	
Affine Collineation (AC)	$\mathbf{A}_{IJ} = x_J \delta^i_I \partial_i$	
Special Projective collineation (SPC)		$\mathbf{P}_I = S_I \mathbf{H}.$

- The Lie point symmetries of all Newtonian dynamical systems are amongst the vectors in the above table. Also the Noether point symmetries of all Newtonian dynamical systems (or systems moving in a flat space in general -apart form some differences in sign) follow from the elements of the first two rows of the above table.

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- Determine all 3d Newtonian dynamical systems which admit Lie symmetries.

Applications in Newtonian Physics

The Lie symmetries of all 3d Newtonian dynamical systems

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Lie symmetry	$\mathbf{F}\left(x_\mu, x_\nu, x_\sigma\right)$
$\frac{d}{2}t\partial_t + \partial_\mu$	$e^{-dx_\mu} f_{\mu,\nu,\sigma}\left(x_\nu, x_\sigma\right)$
$\frac{d}{2}t\partial_t + \partial_{\theta_{(\mu\nu)}}$	$e^{-d\theta_{(\mu\nu)}} f_{\mu,\nu,\sigma}\left(r_{(\mu\nu)}, x_\sigma\right)$
$\frac{d}{2}t\partial_t + R\partial_R$	$x_\mu^{1-d} f_{\mu,\nu,\sigma}\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$
$\frac{d}{2}t\partial_t + x_\mu\partial_\mu$	$x_\mu^{1-d} f_{\mu,\nu,\sigma}\left(x_\nu, x_\sigma\right)$
$\frac{d}{2}t\partial_t + x_\nu\partial_\mu$	$e^{-d\frac{x_\mu}{x_\nu}}\left[\frac{x_\mu}{x_\nu} f_\nu\left(x_\nu, x_\sigma\right) + f_\mu\left(x_\nu, x_\sigma\right)\right]\partial_\mu + f_\nu\partial_\nu + f_\sigma\partial_\sigma$

Applications in Newtonian Physics

The Lie symmetries of all 3d Newtonian dynamical systems

Lie symmetry	$F_\mu\left(x_\mu, x_\nu, x_\sigma\right)$
$t\partial_\mu$	$f_{\mu,\nu,\sigma}\left(x_\nu, x_\sigma\right)$
$t^2\partial_t + tR\partial_R$	$\frac{1}{x_\mu^3} f_{\mu,\nu,\sigma}\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$
$e^{\pm t\sqrt{m}}\partial_\mu$	$-mx_\mu + f_{\mu,\nu,\sigma}\left(x_\nu, x_\sigma\right)$
$\frac{1}{\sqrt{m}}e^{\pm t\sqrt{m}}\partial_t + e^{\pm t\sqrt{m}}R\partial_R$	$-\frac{m}{4}x_\mu + \frac{1}{x_\mu^3} f_{\mu,\nu,\sigma}\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$

- Determine all 3d Newtonian dynamical systems which admit Noether point symmetries and subsequently the ones which are integrable via Noether integrals.

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- Answer

Lie	$d=0$	$d\neq 2$
$\frac{d}{2}t\partial_t+\partial_\mu$	$c_1 x_\mu + f\left(x_\nu, x_\sigma\right)$	$e^{-dx_\mu} f\left(x_\nu, x_\sigma\right)$
$\frac{d}{2}t\partial_t+\partial_{\theta_{(\mu\nu)}}$	$c_1\theta_{(\mu\nu)} + f\left(r_{(\mu\nu)}, x_\sigma\right)$	$e^{-d\theta_{(\mu\nu)}} f\left(r_{(\mu\nu)}, x_\sigma\right)$
$\frac{d}{2}t\partial_t+R\partial_R$	$x^2 f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$	$x^{2-d} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$
$\frac{d}{2}t\partial_t+x_\mu\partial_\mu$	$c_1 x_\mu^2 + f\left(x_\nu, x_\sigma\right)$	$\nexists$
$\frac{d}{2}t\partial_t+x_\nu\partial_\mu$	$c_1 x_\mu + c_2\left(x_\mu^2 + x_\nu^2\right) + f\left(x_\sigma\right)$	$\nexists$

Applications in Newtonian Physics

The Noether symmetries of all 3d Newtonian dynamical systems

Lie	$V(x,y,z)$
$t\partial_\mu$	$c_1 x_\mu + f(x_\nu, x_\sigma)$
$t^2\partial_t + tR\partial_R$	$\frac{1}{x_\mu^2} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$
$e^{\pm t\sqrt{m}}\partial_\mu$	$-\frac{m}{2}x_\mu^2 + c_1 x_\mu + f(x_\nu, x_\sigma)$
$\frac{1}{\sqrt{m}}e^{\pm t\sqrt{m}}\partial_t + e^{\pm t\sqrt{m}}R\partial_R$	$-\frac{m}{8}R^2 + \frac{1}{x_\mu^2} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$

In order a 3d Newtonian dynamical system to be integrable via Noether point symmetries it must admit at least 3 Noether first integrals.

- The same problem for the 2d case has been answered in Tsamparlis M. and Paliathanasis A 2011 J. Phys. A: Math. and Theor. **44** 175202. The 2d case is important because it applies to the mini super space of the dynamical systems in Cosmology.

Motion on the two dimensional sphere

Consider the Lagrangian

$$L\left(\phi,\theta,\dot{\phi},\dot{\theta}\right)=\frac{1}{2}\left(\dot{\phi}^{2}+\mathrm{Sinn}^{2}\phi\ \dot{\theta}^{2}\right)-V\left(\theta,\phi\right) \tag{4}$$

where

$$\mathrm{Sinn}\phi=\begin{cases}\sin\phi & K=1\\ \sinh\phi & K=-1\end{cases}\quad \mathrm{Cosn}\phi=\begin{cases}\cos\phi & K=1\\ \cosh\phi & K=-1.\end{cases}$$

and K is the curvature of the kinetic metric of the Lagrangian (4). The potentials $V\left(\theta,\phi\right)$ where the Dynamical is integrable via Noether point symmetries are the ones of the following table

Motion on the two dimensional sphere

- Integral dynamical systems moving on the 2d Euclidian sphere

$V(\theta,\phi)$	Noether Integral
$F(\cos\theta \mathrm{Sinn}\phi)$	$I_{CK^1_{e,h}}$
$F(\sin\theta \mathrm{Sinn}\phi)$	$I_{CK^2_{e,h}}$
$F(\phi)$	$I_{CK^3_{e,h}}$
$F\left(\frac{1+\tan^2\theta}{\mathrm{Sinn}^2\phi\ (a-b\tan\theta)^2}\right)$	$aI_{CK^1_{e,h}}+bI_{CK^2_{e,h}}$
$F(a\cos\theta \mathrm{Sinn}\phi - K\ b\mathrm{Cosn}\phi)$	$aI_{CK^1_{e,h}}+bI_{CK^3_{e,h}}$
$F(a\sin\theta \mathrm{Sinn}\phi - K\ b\mathrm{Cosn}\phi)$	$aI_{CK^2_{e,h}}+bI_{CK^3_{e,h}}$
$F\left(\begin{array}{c}(a\cos\theta - b\sin\theta)\,\mathrm{Sinn}\phi+ \\ -K\ c\mathrm{Cosn}\phi\end{array}\right)$	$aI_{CK^1_{e,h}}+bI_{CK^2_{e,h}}+cI_{CK^3_{e,h}}$

where

$$\begin{aligned} I_{CK^3} &= \dot\theta \mathrm{Sinn}^2\phi.\ ,\quad I_{CK^1} = \dot\phi\sin\theta + \dot\theta\cos\theta \mathrm{Sinn}\phi \mathrm{Cosn}\phi \\ I_{CK^2} &= \dot\phi\cos\theta - \dot\theta\sin\theta \mathrm{Sinn}\phi \mathrm{Cosn}\phi \end{aligned}$$

Motion on the two dimensional sphere

Corollary

A dynamical system with Lagrangian (4) has one, two or four Noether point symmetries hence Noether integrals.

Proof.

For the case of the free particle we have the maximum number of four Noether symmetries (the rotation group $so(3)$ plus the ∂_t). In the case the potential is not constant the Noether symmetries are produced by the non-gradient KVs with Lie algebra $[X_A, X_B] = C^C_{AB} X_C$ where $C^3_{12} = C^2_{31} = C^1_{23} = 1$ for $\varepsilon = 1$ and $\bar{C}^3_{21} = \bar{C}^1_{23} = \bar{C}^2_{31} = 1$ for $\varepsilon = -1$. Because the Noether point symmetries form a Lie algebra and the Lie algebra of the KVs is semisimple the system will admit either none, one or three Noether symmetries generated from the KVs. The case of three is when $V(\theta, \phi) = V_0$ that is the case of geodesics, therefore the Noether point symmetries will be (including ∂_t) either one, two or four. $\square$

Lie and Noether symmetries of Bianchi class A homogeneous cosmologies with a scalar field.

- The Bianchi models in the ADM formalism are described by the metric
$$ds^2 = -N^2(t)dt^2 + g_{\mu\nu}\omega^{\mu} \otimes \omega^{\nu} \tag{5}$$
where $N(t)$ is the lapse function and $\{\omega^a\}$ is the canonical basis 1-forms which satisfy the Lie algebra $d\omega^i = C^i_{jk}\omega^j \wedge \omega^k C^i_{jk}$ being the structure constants of the algebra.

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- The spatial metric $g_{\mu\nu}$ splits so that $g_{\mu\nu} = \exp(2\lambda)\exp(-2\beta)_{\mu\nu}$ where $\exp(2\lambda)$ is the scale factor of the universe and $\beta_{\mu\nu}$ is a 3×3 symmetric, traceless matrix, which can be written in a diagonal form with two independent quantities, the anisotropy parameters β_+, β_-, as follows:

$$\beta_{\mu\nu} = diag\left(\beta_+, -\frac{1}{2}\beta_+ + \frac{\sqrt{3}}{2}\beta_-, -\frac{1}{2}\beta_+ - \frac{\sqrt{3}}{2}\beta_-\right). \tag{6}$$

Lagrangian description

- The Lagrangian leading to the full Bianchi scalar field dynamics is

$$L = e^{3\lambda}\left[R^{*} + 6\lambda - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2) - \dot{\phi}^2 + V(\phi)\right] \tag{7}$$

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- R^{*} is the Ricci scalar of the 3 dimensional spatial hypersurfaces given by the expression:

$$R^{*} = -\frac{1}{2}e^{-2\lambda}\left[N_1^2 e^{4\beta_1} + e^{-2\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2}\right)^2 - 2N_1 e^{\beta_1}\left(N_2\right.\right.$$

$$+\frac{1}{2}N_1 N_2 N_3(1 + N_1 N_2 N_3).$$

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$$+\frac{1}{2}N_1 N_2 N_3(1 + N_1 N_2 N_3).$$

- The constants N_1, N_2, and N_3 are the components of the classification vector n^μ and $\beta_1 = -\frac{1}{2}\beta_+ + \frac{\sqrt{3}}{2}\beta_-$, $\beta_2 = -\frac{1}{2}\beta_+ - \frac{\sqrt{3}}{2}\beta_-$.
It is important to note that the curvature scalar R^* does not depend on the derivatives of the anisotropy parameters β_+, β_- , equivalently of β_1, β_2.

- The Euler Lagrange equations due to the Lagrangian (7) are:

$$\begin{aligned}
\ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 + \frac{3}{8}(\dot{\beta}_1^2 + \dot{\beta}_2^2) + \frac{1}{4}\dot{\phi}^2 - \frac{1}{12}e^{-3\lambda}\frac{\partial}{\partial\lambda}\left(e^{3\lambda}R^*\right) - \frac{1}{2}V(\phi) &= 0 \\
\ddot{\beta}_1 + 3\dot{\lambda}\dot{\beta}_1 + \frac{1}{3}\frac{\partial R^*}{\partial \beta_1} &= 0 \\
\ddot{\beta}_2 + 3\dot{\lambda}\dot{\beta}_2 + \frac{1}{3}\frac{\partial R^*}{\partial \beta_2} &= 0 \\
\ddot{\phi} + 3\dot{\phi}\dot{\lambda} + \frac{\partial V}{\partial \phi} &= 0
\end{aligned}$$

where a dot over a symbol indicates derivative with respect to t.

- The Euler Lagrange equations due to the Lagrangian (7) are:

$$\ddot{\lambda}+\frac{3}{2}\dot{\lambda}^{2}+\frac{3}{8}(\dot{\beta}_{1}^{2}+\dot{\beta}_{2}^{2})+\frac{1}{4}\dot{\phi}^{2}-\frac{1}{12}e^{-3\lambda}\frac{\partial}{\partial\lambda}\left(e^{3\lambda}R^{*}\right)-\frac{1}{2}V(\phi)=0$$

$$\ddot{\beta}_{1}+3\dot{\lambda}\dot{\beta}_{1}+\frac{1}{3}\frac{\partial R^{*}}{\partial\beta_{1}}=0$$

$$\ddot{\beta}_{2}+3\dot{\lambda}\dot{\beta}_{2}+\frac{1}{3}\frac{\partial R^{*}}{\partial\beta_{2}}=0$$

$$\ddot{\phi}+3\dot{\phi}\dot{\lambda}+\frac{\partial V}{\partial\phi}=0$$

  where a dot over a symbol indicates derivative with respect to t.

- We apply Theorem 1 and Theorem 2 in order to compute the Lie and the Noether symmetries of class A Bianchi models.
  Similar incomplete works on that topic Cotsakis S et. al. 1998 Grav. Cosm. 4 314 , Capozzielo S et.al. 1997 J. Mod Phys. D 6 491, Vakili B et. al. 2007 Class. Quantum Grav. 24 931.

- We consider the four dimensional Riemannian space with coordinates $x^i = (\lambda, \beta_1, \beta_2, \phi)$ and metric

$$ds^2 = e^{3\lambda}\left(12d\lambda^2 - 3d\beta_1^2 - 3d\beta_2^2 - 2d\phi^2\right). \tag{8}$$

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- The metric is the conformally flat FRW spacetime whose special projective algebra consists of the non gradient KVs

$$Y^1 = \partial_{\beta_1},\ Y^2 = \partial_{\beta_2},\ Y^3 = \partial_\phi,\ Y^4 = \beta_2\partial_{\beta_1} - \beta_1\partial_{\beta_2}$$
$$Y^5 = \phi\partial_{\beta_1} - \frac{3}{2}\beta_1\partial_\phi,\ Y^6 = \phi\partial_{\beta_2} - \frac{3}{2}\beta_2\partial_\phi$$

and the gradient HV

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- The Lagrangian is written $L = T - U(x^i)$ where $T = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^i$ is the geodesic Lagrangian, the potential function is

$$U(x^i) = -e^{3\lambda}\left(V\left(\phi\right) + R^*\right) \qquad (9)$$

and we have used the fact that the curvature scalar does not depend on the derivatives of the coordinates β_1, β_2.

- We apply Theorem 1 and Theorem 2 to determine the Lie and the Noether symmetries of the dynamical system with Lagrangian (7).

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- We determine the Lie and the Noether symmetries in the following cases:

  Case 1. Vacuum. In this case $\phi =$constant.

  Case 2. Zero potential $V(\phi) = 0$, $\dot{\phi} \neq 0$

  Case 3. Constant Potential $V(\phi) =$constant, $\dot{\phi} \neq 0$

  Case 4. Arbitrary Potential $V(\phi)$, $\dot{\phi} \neq 0$.

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- The results for Bianchi I and Bianchi II and Bianchi VI/VIIare shown in the following tables

Bianchi I

Bianchi I	Noether Sym.	Lie Sym.
Vacuum	$\partial_t,\ Y^1,\ Y^2,\ Y^4$	$\partial_t,\ t\partial_t,\ Y^{1,2,4},\ H^i$
	$2t\partial_t + H^i,\ t^2\partial_t + tH^i$	$t^2\partial_t + tH^i$
Zero Pot.	$\partial_t, Y^{1-6}\ ,\ 2t\partial_t + H^i$	$\partial_t,\ t\partial_t,\ Y^{1-6}$
	$t^2\partial_t + tH^i$	$H^i,\ t^2\partial_t + tH^i$
Constant Pot.	$\partial_t,\ Y^{1-6}$	$\partial_t,\ Y^{1-6}, H^i$
	$\frac{1}{C}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$	$\frac{1}{C}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$
Arbitrary Pot.	$\partial_t,\ Y^{1,2,4}$	$\partial_t,\ Y^{1,2,4}\ , H^i$
Exponential Pot.	$\partial_t,\ Y^{1,2,4}$	$\partial_t,\ Y^{1,2,4}\ , H^i$
	$2t\partial_t + H^i + \frac{4}{d}Y^3$	$t\partial_t + \frac{2}{d}Y^3$

Bianchi II	Noether Sym.	Lie Sym.
Vacuum	$\partial_t,\ Y^2$	$\partial_t,\ Y^2$
	$6t\partial_t + 3H^i - 5Y^1$	$\frac{1}{3}t\partial_t + H^i,\ t\partial_t - Y^1$
Zero Pot.	$\partial_t,\ Y^2,\ Y^3,\ Y^6$	$\partial_t,\ Y^2,\ Y^3,\ Y^6$
	$6t\partial_t + 3H^i - 5Y^1$	$\frac{1}{3}t\partial_t + H^i,\ t\partial_t - Y^1$
Constant Pot.	$\partial_t,\ Y^2,\ Y^3,\ Y^6$	$\partial_t,\ Y^2,\ Y^3,\ Y^6$
		$3H^i + Y^1$
Arbitrary Pot.	$\partial_t,\ Y^2$	$\partial_t,\ Y^2,\ 3H^i + Y^1$
Exponential Pot.	$\partial_t,\ Y^2$	$\partial_t,\ Y^2,\ 3H^i + Y^1$
	$2t\partial_t + H^i - \frac{5}{3}Y^1 + \frac{4}{d}Y^3$	$t\partial_t + \frac{2}{d}Y^3$

Bianchi VI/VII

Bianchi $\mathbf{VI}_0$ / $\mathbf{VII}_0$	Noether Sym.	Lie Sym.
Vacuum	$\partial_t, \;\; 6t\partial_t + 3H^i +$	$\partial_t, \;\; H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2$
	$-2Y^1 - 2\sqrt{3}Y^2$	$2t\partial_t - Y^1 - \sqrt{3}Y^2$
Zero Pot.	$\partial_t, \;\; Y^3, 6t\partial_t + 3H^i +$	$\partial_t, \;\; H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2$
	$-2Y^1 - 2\sqrt{3}Y^2$	$Y^3, 2t\partial_t - Y^1 - \sqrt{3}Y^2$
Constant Pot.	$\partial_t, \;\; Y^3$	$\partial_t, \;\; Y^3, \;\; H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}$
Arbitrary Pot.	∂_t	$\partial_t, \;\; H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2$
Exponential Pot.	$\partial_t \;, \;\; 6t\partial_t + 3H^i - 2Y^1 +$	$\partial_t, \;\; H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2$
	$-2\sqrt{3}Y^2 + \frac{6}{d}Y^3$	$t\partial_t + \frac{1}{d}Y^3$

Bianchi VIII / IX

Bianchi VIII	Noether Sym.	Lie Sym.
Vacuum	∂_t	$\partial_t, \ \frac{2}{3}t\partial_t \ + H^i$
Zero Pot.	$\partial_t, \ Y^3$	$\partial_t, \ Y^3, \ \frac{2}{3}t\partial_t \ + H^i$
Constant Pot.	$\partial_t, \ Y^3$	$\partial_t, \ Y^3$
Arbitrary Pot.	∂_t	∂_t
Bianchi IX	Noether Sym.	Lie Sym.
Vacuum	∂_t	∂_t
Zero Pot.	$\partial_t, \ Y^3$	$\partial_t, \ Y^3$
Constant Pot.	$\partial_t, \ Y^3$	$\partial_t, \ Y^3$
Arbitrary Pot.	∂_t	∂_t

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