# Decoupling of Field Equations in Einstein and Modified Gravity

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# Introduction and Goals

**Motivation:** Accelerating universe, dark energy/matter, QG:

$\nabla[\mathbf{g}] \rightarrow \mathbf{D}[\mathbf{g}]$, $R \rightarrow f(R, T, ...)$ Modified Gravity, MG, and Field Eqs

**Goal:** Geometric Method for constructing "exact" solutions.

2+2 decomposition & auxiliary connections:
Decoupling property in nonholonomic frames
Formal Integration: exact solutions in very general forms

generic off-diagonal metrics & auxiliary, or Levi-Civita, connections;
generating and integration functions, all coordinates, parameters.

**Criteria:** Effective Lorenz manifolds via nonholonomic deformations;

the Cauchy problem; asymptotic conditions, GR limits.
**"Orthodox philosophy" for GR:** MG as an effective GR with off–diagonal nonlinear interactions and nonholonomic constraints.

**Examples:** (Black) ellipsoids and solitons in MG

# Outline

# Modified gravity and Ricci solitons

**Modifications of GR:** $\nabla[\mathbf{g}] \to \mathbf{D}[\mathbf{g}]$; Lagrange density $R \to f(R,T)$

Vacuum MG $\quad f_R \mathbf{R}_{\alpha\beta} - \frac{1}{2} f \mathbf{g}_{\alpha\beta} + (\mathbf{g}_{\alpha\beta} \mathbf{D}_\gamma \mathbf{D}^\gamma - \mathbf{D}_\alpha \mathbf{D}_\beta) f_R = 0,$

for $f_R = \partial f / \partial R$. If $\mathbf{D} = \nabla$, vacuum $f(R)$ gravity.

generalized Ricci solitons: $\quad \mathbf{R}_{\alpha\beta} + \mathbf{D}_\alpha \mathbf{D}_\beta K = \lambda \mathbf{g}_{\alpha\beta},$

$K = f_R$ and $\mathbf{D} \to \nabla$ and $\mathbf{g} \to \widehat{\mathbf{g}}$; stationary geometric flows; generalized Einstein spaces; bridge to QG.

MG with effective Newton/ cosmological "constants" and field eqs

$$\mathbf{R}_{\alpha\beta} = \Lambda(x^i, y^a) \mathbf{g}_{\alpha\beta},$$

"Polarized" cosmological constant $\quad \Lambda = \frac{\lambda + \mathbf{D}_\gamma \mathbf{D}^\gamma f_R - f/2}{1 - f_R}.$

Generic off–diagonal solutions generated in explicit form for Killing symmetry, on $\partial / \partial y^4$ (for simplicity), when $\Lambda \approx \Lambda(x^i)$.

# Decoupling in MG and GR

## Nonholonomic 2+2 splitting

**Aim:** Find $e_\alpha = e_\alpha^{\alpha'} \partial_{\alpha'}$, parametrize metrics & connections $\rightarrow$ MG field eqs decouple; integrate with generic off–diagonal metrics depending on all coordinates.

### Non–integrable 2+2 splitting on $(V, \mathbf{g})$

indices $\alpha, \beta, ... = (i, a), (j, b), ...$ for $i, j, k, ... = 1, 2$; $a, b, c, .. = 3, 4$
coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3, y^4)$, or $u = (x, y)$,
partial derivatives $\partial_\alpha := \partial / \partial u^\alpha$; $\partial_\alpha = (\partial_i, \partial_a)$

$$\begin{aligned} \mathbf{N} : TV &= hTV \oplus vTV \\ \mathbf{N} &= N_i^a(x, y) \partial_a \otimes dx^i \end{aligned}$$

N–adapted frames:
$$\begin{aligned} \mathbf{e}_\alpha &:= (\mathbf{e}_i = \partial_i - N_i^a \partial_a, e_b = \partial_b) \\ \mathbf{e}^\beta &:= (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i) \end{aligned}$$

Nonholonomic:
$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = \mathbf{w}^\gamma{}_{\alpha\beta}(u)\, \mathbf{e}_\gamma$$

# Decoupling in MG and GR

**N–adapted,** $\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} = g_{ij}(x,y)\, e^i \otimes e^j + g_{ab}(x,y)\mathbf{e}^a \otimes \mathbf{e}^b$

Coordinate bases: $\mathbf{g} = \underline{g}_{\alpha\beta}(u)\, du^{\alpha} \otimes du^{\beta}$, $\quad \underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e\, g_{be} & g_{ab} \end{bmatrix}$

$N_i^a \neq A_{bi}^a(x)y^b$, not Kaluza–Klein gravity.

**Remark:** $\exists$ other metric compatible $\mathbf{D}$ completely defined by $\mathbf{g}$

"auxiliary" N–adapted connection $\mathbf{D}$: 1) $\mathbf{Dg} = 0$, 2) $T^i_{\ jk} = 0, T^a_{\ bc} = 0, \exists T^a_{\ jk} \neq 0$
not–N–adapted Levi–Civita, $\nabla\mathbf{g} = 0$, zero torsion; $\nabla = \mathbf{D} + \mathbf{Z}[\mathbf{T}]$,

**IDEA:** GR and MG can be formulated equivalently in terms of $\nabla$ and/or $\mathbf{D}$;
chose $\mathbf{D}$ to decouple and integrate field equations; at the end fix zero torsion if necessary.

**Ansatz:** $g_{\alpha\beta} = diag[g_i(x^k), h_a(x^k, y^3 := v)],\ N_i^3 = w_i(x^k, v),\ N_i^4 = n_i(x^k, v), \varepsilon_i = \pm 1$,

$$
\begin{aligned}
{}^K\mathbf{g} &= \varepsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a, \\
\mathbf{e}^3 &= dv + w_i(x^k, v)dx^i,\ \mathbf{e}^4 = dy^4 + n_i(x^k, v)dx^i.
\end{aligned}
$$

# Decoupling in MG and GR

**Theorem 1 (Decoupling):** effective Einstein eqs for ${}^{K}\mathbf{g}$ and $\Lambda(x^i, \theta)$,

with $a^{\bullet} = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial v$), parameters $\theta$, for $h^*_{3,4} \neq 0, \Lambda \neq 0$, are

$$\begin{aligned} \varepsilon_1 \ddot{\psi} + \varepsilon_2 \psi'' &= 2\Lambda \\ \phi^*\, h_4^* &= 2 h_3 h_4 \Lambda \\ \beta w_i + \alpha_i &= 0 \\ n_i^{**} + \gamma n_i^* &= 0 \end{aligned}$$

for $\alpha_i = h_4^* \partial_i \phi, \beta = h_4^*\, \phi^*, \gamma = \left(\ln \frac{|h_4|^{3/2}}{|h_3|}\right)^*$

generating function $\phi = \ln |\frac{h_4^*}{\sqrt{|h_3 h_4|}}|$

**Remarks:** 1) we can not "see" decoupling for the LC in non–N–adapted frames.

2) $\exists$ decoupling for non–Killing ansatz and $h_3^* = 0$, or $h_4^* = 0$

# Theorem 2 (Integral Varieties)

**Theorem 2** (constructing off–diagonal "one–Killing" solutions)

$$g_i = \varepsilon_i e^{\psi},$$

$$h_3 = {}^0h_3 \left[1 + (e^{\phi})^*/2\Lambda\sqrt{|{}^0h_3|}\right]^2, \; h_4 = {}^\circ h_4 \exp[e^{2\,\phi}/8\Lambda]$$

$$w_i = -\partial_i \phi/\phi^*$$

$$n_k = {}_1n_k + {}_2n_k \int [h_3/(\sqrt{|h_4|})^3] dv$$

generating functions $\psi(x^k, \theta), \phi(x^k, v, \theta)$; source $\Lambda(x^k, \theta)$,
integration functions ${}^0h_a(x^k, \theta),\ {}_1n_k(x^k, \theta),\ {}_2n_k(x^k, \theta)$

"slight violation" of decoupling for the LC conditions

$$w_i^* = \mathbf{e}_i \ln|h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k, \qquad n_i^* = 0,\ \partial_i n_k = \partial_k n_i \to\ {}_2n_i = 0.$$

## Dependence on $y^4$, "vertical" conformal $\omega^2(x^j, v, y^4), \partial a/\partial y^4 := a^\circ$,

$\omega^2 = 1$ results in solutions with Killing symmetry,

$$\mathbf{g} = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j, v, y^4)h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a,$$
$$\mathbf{e}^3 = dy^3 + w_i(x^k, v)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, v)dx^i,$$
$$\mathbf{e}_k\omega = \partial_k\omega + w_k\omega^* + n_k\omega^\circ = 0.$$

## N–deformations & gravitational polarizations $\eta_\alpha,\ \eta_i^a$,

N–deforms, $\ ^\star\mathbf{g} = [\ ^\star g_i,\ ^\star h_a,\ ^\star N_k^a] \rightarrow\ ^\eta\mathbf{g} = [\ g_i, h_a, N_k^a]$,

$$^\eta\mathbf{g} = \eta_i(x^k, v)\ ^\star g_i(x^k, v)dx^i \otimes dx^i + \eta_a(x^k, v)\ ^\star h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a,$$

$$\mathbf{e}^3 = dv + \eta_i^3(x^k, v)\ ^\star w_i(x^k, v)dx^i,\ \mathbf{e}^4 = dy^4 + \eta_i^4(x^k, v)\ ^\star n_i(x^k, v)dx^i.$$

For a solution in GR with well–defined boundary/ asymptotic conditions, we can search $\ ^\star\mathbf{g} \rightarrow\ ^\eta\mathbf{g}$
to a "parametric/noncommutative/stochastic ..." solution in GR, MG.

# Cauchy Problem & Decoupling Property

Choquet–Bruhat, 1952: $\exists$ a set of hyperbolic equations underlying the Einstein eqs with $\Lambda$ .
Mathematical Relativity: D. Christodoulou, S. Klainerman, I. Rodnianski, P. Chruściel ...

**Goal:** study the evolution (Cauchy) problem in N–adapted form and the decoupling in GR & MG.

**Definition (N–adapted wave coordinates):** A set $\{\widehat{u}^{\mu} = (\widehat{x}^{i}, \widehat{y}^{a})\}$ is canonically N–harmonic, i.e. it both harmonic and adapted to a splitting **N** if $\widehat{\square}\widehat{u}^{\mu} = 0$, where d'Alambert operator $\widehat{\square} := \widehat{\mathbf{D}}_{\alpha}\ \widehat{\mathbf{D}}^{\alpha}$ acts on $f(x,y)$,

$$\begin{aligned}\widehat{\square} f &:= \left(\sqrt{|\mathbf{g}_{\alpha\beta}|}\right)^{-1}\mathbf{e}_{\mu}\left(\sqrt{|\mathbf{g}_{\alpha\beta}|}\mathbf{g}^{\mu\nu}\mathbf{e}_{\nu} f\right) \\ &= \left(\sqrt{|g_{kl}|}\right)^{-1}\mathbf{e}_{i}\left(\sqrt{|g_{kl}|}g^{ij}\mathbf{e}_{j} f\right) + \left(\sqrt{|g_{cd}|}\right)^{-1}\partial_{a}\left(\sqrt{|g_{kl}|}g^{ab}e_{b} f\right).\end{aligned}$$

**Lemma:** In canonical N–harmonic coordinates, the (modified) Einstein eqs with $\Lambda$ are

$$\widehat{\mathbf{E}}^{\alpha\beta} = \widehat{\square}\mathbf{g}^{\alpha\beta} - \mathbf{g}^{\tau\theta}\left[(\mathbf{g}^{\alpha\mu}\widehat{\Gamma}^{\beta}{}_{\mu\nu} + \mathbf{g}^{\alpha\mu}\widehat{\Gamma}^{\beta}{}_{\mu\nu})\widehat{\Gamma}^{\nu}{}_{\tau\theta} + 2\mathbf{g}^{\gamma\mu}\widehat{\Gamma}^{\alpha}{}_{\mu\theta}\widehat{\Gamma}^{\beta}{}_{\tau\gamma}\right] - 2\Lambda\mathbf{g}^{\alpha\beta} = 0;$$

such PDE for $\mathbf{g}^{\alpha\beta}$ form a system of 2d order quasi–linear N–adapted wave–type eqs.

# Cauchy Problem

## Theorem on N–adapted solutions

We can apply the standard theory of hyperbolic PDE.
Let $H^k_{loc}$ are the Sobolev spaces of functions which are in $L^2(K)$ for any compact set $K$ when their distributional derivatives are considered up to an integer order $k$ also in $L^2(K)$.
We use N–adapted wave coordinates with additional formal $3+1$ splitting, $\widehat{u}^{\mu} = ({}^t\widehat{u}, \widehat{u}^{\bar{i}})$.

Standard results from the PDE theory $\rightarrow$

**Theorem:** The MG field eqs with $\Lambda$ have a unique solution $\mathbf{g}^{\alpha\beta}$

defined by N–adapted PDE stated on an open neighborhood $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^3$ of $\mathcal{O} \subset \{0\} \times \mathbb{R}^3$ with any initial data

$$\mathbf{g}^{\alpha\beta}(0, \widehat{u}^{\bar{i}}) \in H^{k+1}_{loc} \text{ and } \frac{\partial \mathbf{g}^{\alpha\beta}(0, \widehat{u}^{\bar{i}})}{\partial({}^t\widehat{u})} \in H^{k+1}_{loc}, k > 3/2.$$

The set $\mathcal{U}$ can be chosen in a form that $(\mathcal{U}, \mathbf{g}^{\alpha\beta})$ is globally hyperbolic with Cauchy surface $\mathcal{O}$.

On initial data sets and global nonholonomic evolution, see arXiv: 1108.2022

# Black Ellipsoids?

$x^1 = \xi = \int dr/|\underline{q}(r)|^{1/2}, x^2 = \vartheta = r(\xi)\theta, y^3 = \varphi, y^4 = t; \underline{q}(r) = 1 - 2\frac{m(r)}{r} - \lambda\frac{r^2}{3}, \lambda = const$

$$^{\circ}\mathbf{g} = d\xi \otimes d\xi + r^2(\xi)\ d\theta \otimes d\theta + r^2(\xi)\sin^2\theta\ d\varphi \otimes d\varphi - \underline{q}(\xi)\ dt \otimes\ dt,$$

(ellipsoid) de Sitter rotoid configurations with small eccentricity $\varepsilon$:

$$\begin{aligned} {}^{rot}_{\lambda}\mathbf{g} &= e^{\psi(\xi,\vartheta)}\,(d\xi^2 +\ d\vartheta^2) \\ &+ r^2(\xi)\sin^2\theta(1 + \frac{\partial_\varphi e^\phi}{2\Lambda\sqrt{|\underline{q}|}})^2\ \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi - [\underline{q}(\xi) + \varepsilon\zeta(\xi,\vartheta,\varphi)\mathbf{e}_t \otimes \mathbf{e}_t, \\ \mathbf{e}_\varphi &= d\varphi - \frac{\partial_\xi \phi}{\partial_\varphi \phi}d\xi - \frac{\partial_\vartheta \phi}{\partial_\varphi \phi}d\vartheta,\ \mathbf{e}_t = dt + n_1(\xi,\vartheta,\varphi)d\xi + n_2(\xi,\vartheta,\varphi)d\vartheta, \end{aligned}$$

Generating function $\quad e^{2\phi} = 8\Lambda \ln|1 - \varepsilon\zeta/\underline{q}(\xi)|,\ \zeta =\ \underline{\zeta}(\xi,\vartheta)\sin(\omega_0\varphi + \varphi_0),$

$$\text{the term before } \mathbf{e}_t \otimes \mathbf{e}_t,\ h_4 = 0,\ \text{horizon } r_+ \simeq \frac{2\ m_0}{1 + \varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)}$$

Small deformations on parameter $\varepsilon$, $h_3 =\ \check{h}_3(1 + \varepsilon\chi_3), h_4 =\ \check{h}_4(1 + \varepsilon\chi_4), w_i \sim \varepsilon\check{w}_i, n_i \sim \varepsilon\check{n}$

# Modified Solitonic Configurations

non–stationary ansatz

$$ds^2 = e^{\psi(\xi,\vartheta)}[d\xi^2 + d\vartheta^2] - \underline{q}(\xi)(1 + \frac{\partial_t e^{\phi(\xi,\vartheta,t)}}{2\Lambda(\xi,\vartheta)\sqrt{|\underline{q}(\xi)|}})^2[dt - \frac{\partial_\xi \phi}{\partial_t \phi} d\xi - \frac{\partial_\vartheta \phi}{\partial_t \phi} d\vartheta]^2$$

$$+ r^2(\xi)\sin^2\vartheta \exp[\frac{e^{2\phi(\xi,\vartheta,t)}}{8\Lambda(\xi,\vartheta)}]\,[d\varphi + \left( {}^1n_1(\xi,\vartheta) + {}^2n_1(\xi,\vartheta) \int dt \frac{h_3(\xi,\vartheta,t)}{(\sqrt{|h_4(\xi,\vartheta,t)|})^3} \right) d\xi$$

$$+ \left( {}^1n_2(\xi,\vartheta) + {}^2n_2(\xi,\vartheta) \int dt \frac{h_3(\xi,\vartheta,t)}{(\sqrt{|h_4(\xi,\vartheta,t)|})^3} \right) d\vartheta]^2,$$

coordinates $x^1 = \xi, x^2 = \vartheta, y^3 = t, y^4 = \varphi$ and $\underline{q}(\xi) = \underline{q}(r(\xi))$

**Solitonic backgrounds with radial Burgers equation:** $\phi(\xi,\vartheta,t) = \eta(\xi,\vartheta,t)$, $y^4 = t$,

$$\text{KdP equation,} \quad \pm\,\eta^{''} + (\partial_t \eta + \eta\,\eta^{\bullet} + \epsilon \eta^{\bullet\bullet\bullet})^{\bullet} = 0,$$

The dispersionless limit $\epsilon \to 0$ independent on $x^2$: Burgers' equation $\partial_t \eta + \eta\,\eta^{\bullet} = 0$.

# Conclusions & Perspectives

- a) "∀" solution of Einstein eqs, $g_{\alpha'\beta'}$, via $e_\alpha = e_\alpha^{\alpha'}(x^i, y^a)e_{\alpha'}$, $\mathbf{g}_{\alpha\beta} = e_\alpha^{\alpha'} e_\beta^{\beta'} g_{\alpha'\beta'}, \rightarrow$

$$\mathbf{g}_{\alpha\beta} = \left| \begin{array}{cccc} g_1 + \omega^2({w_1}^2 h_3 + n_1^2 h_4) & \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & \omega^2\ w_1 h_3 & \omega^2\ n_1 h_4 \\ \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & g_2 + \omega^2({w_2}^2 h_3 + n_2^2 h_4) & \omega^2\ w_2 h_3 & \omega^2\ n_2 h_4 \\ \omega^2\ w_1 h_3 & \omega^2\ w_2 h_3 & \omega^2 h_3 & 0 \\ \omega^2\ n_1 h_4 & \omega^2\ n_2 h_4 & 0 & \omega^2 h_4 \end{array} \right|$$

b) "Very general" ansatz
$$\begin{aligned} \mathbf{g} &= g_i dx^i \otimes dx^i + \omega^2 h_a \underline{h}_a \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + (w_i + \underline{w}_i)\, dx^i,\ \mathbf{e}^4 = dy^4 + (n_i + \underline{n}_i)\, dx^i, \end{aligned}$$

$$\begin{aligned} g_i &= g_i(x^k), g_a = \omega^2(x^i, y^c)\ h_a(x^k, y^3)\underline{h}_a(x^k, y^4),\ \text{not summation on "a"}, \\ N_i^3 &= w_i(x^k, y^3) + \underline{w}_i(x^k, y^4), N_i^4 = n_i(x^k, y^3) + \underline{n}_i(x^k, y^4), \end{aligned}$$

are functions of necessary smooth class generating solutions of Einstein eqs.

- For well defined conditions, the gravitational field eqs in MG can be considered as effective Ricci soliton and/or Einstein eqs with polarized cosmological constants. It is possible to construct general off–diagonal integrals of such solutions... Formal integration...
- There are examples of black ellipsoid and solitonic configurations which may survive as generic off–diagonal limits to GR. Concept of general solutions? Cauchy problem? Asymptotic conditions? Boundary conditions...symmetries....