Asymptotically AdS spacetimes and isometric embeddings

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June 22, 2012

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Hyperboloid model of H^n

Hyperbolic space H^n s a Riemannian manifold. All of the geometry can be understood in terms of intrinsic properties (metric, curvature..). However, to manifest more symmetries consider: ($\mathbb{M}^{1,n}$ is Minkowski space-time)

$$\mathcal{H}^n := \{X^\mu \in \mathbb{M}^{1,n} | X^0 > 0,\ X^\mu X^\nu \eta_{\mu\nu} = -1\}. \tag{1}$$

The intrinsic geometry of this hyperboloid is equivalent (globally isometric) to H^n.

- Every timelike straight line through the origin intersects a unique point in $\mathcal{H}$.
- Geodesics: $\mathcal{H} \cap$ timelike 2-planes through the origin.
- Totally geodesic p-surfaces: $\mathcal{H} \cap$ timelike $(p+1)$-planes through the origin.
- Global isometry group: $PSO(n, 1)$.

Hyperboloid model of $\widetilde{\text{AdS}}^n$

Submanifold of $\mathbb{M}^{2,n-1}$ (flat spacetime with two time coordinates) . Define $\eta^{(2,n-1)}_{\mu\nu} := \text{diag}(-1, +1, ..., +1, -1)$.

$$\mathcal{A} := \{X^\mu \in \mathbb{M}^{2,n-1} | X^\mu X^\nu \eta^{(2,n-1)}_{\mu\nu} = -1\}. \quad (2)$$

The intrinsic geometry of this hyperboloid is Anti de Stter space $\widetilde{\text{AdS}}^n$. This has closed timelike curves.

- Every timelike straight line through the origin intersects two unique points in $\mathcal{H}$.
- $\mathcal{A}\cap$ [2-plane through the origin] are geodesics (timelike) or a pair of disconnected geodesics (spacelike)
- Global isometry group: $SO(n, 1)$.

The cosmological AdS space is the universal cover of $\widetilde{\text{AdS}}$. Later we present an embedding of AdS^n into $\mathbb{M}^{2,n}$.

Definition (Embedding of differentiable manifold)

A differentiable embedding $\phi : M \to N$ is an injective map (for x, y distinct $\phi(x) \neq \phi(y)$) so that the image $\phi(M)$ is homeomorphic to M and furthermore the induced tangent space map is injective.

Note $dim(N) \geq dim(N)$. In terms of coordinates (σ^a) on M and (x^μ) on N, we write $\phi : \sigma \to x(\sigma)$ and $\phi_* : v^a \to v^a \partial x^\mu / \partial \sigma^a$.

Definition (isometric embedding)

Let (M, h) and (N, g) be (pseudo-)Riemannian manifolds. A smooth embedding $\phi : M \to N$ is isometric if $\phi_* g = h$.

The BTZ black hole ($J = 0$)

In 2+1 dimensional gravity with negative cosmological constant, the solutions satisfy $R^{\mu\nu}{}_{\kappa\lambda} = -\frac{1}{l^2}\delta^{\mu\nu}_{\kappa\lambda}$. We set $l = 1$. The spherically symmetric solution has the static form

$$ds^2 = (r^2 - a^2)d\tau^2 + \frac{dr^2}{r^2 - a^2} + r^2 d\phi^2$$

outside of the event horizon ($r = a$). Kruskal type coordinate system:

$$ds^2 = 4\frac{-dt^2 + dx^2}{(1 + t^2 - x^2)^2} + a^2\frac{(1 - t^2 + x^2)^2}{(1 + t^2 - x^2)^2}d\phi^2 .$$

The domain of the coordinates is $-1 < -t^2 + x^2 < 1$, $\phi \sim \phi + 2\pi$. Singularities at $t^2 - x^2 = 1$, conformal infinity at $x^2 - t^2 = 1$, event horizons $x = \pm t$, bifurcation surface at $x = t = 0$. This covers the maximally extended space-time.

Lemma (S.W. arXiv:1011.3883 gr-qc)

The nonrotating BTZ black hole spacetime can be globally isometrically embedded into the region $X^0 > 0$ of $\mathbb{M}^{2,3}$. The image is the intersection of quadric hypersurfaces:

$$(X^1)^2 + (X^2)^2 = \frac{a^2}{1+a^2}(X^0)^2\,, \quad (X^3)^2 - (X^4)^2 = -1 + \frac{1}{1+a^2}(X^0)^2$$

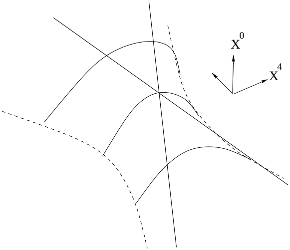
The past and future singularities are located at the intersection of the two constraint surfaces with the hyperplane $X^0 = 0$.

Proof: It can be verified that the following is an embedding

$$X^0(x,t) = \sqrt{1+a^2}\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\,,$$

$$X^1(x,t,\phi) = a\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\cos\phi\,,\; X^2(x,t,\phi) = a\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\sin\phi\,,$$

$$X^3(x,t) = \frac{2x}{1+t^2-x^2}\,,\; X^4(x,y) = \frac{2t}{1+t^2-x^2}\,.$$



- By lifting the restriction $X^0 > 0$ we obtain two copies of BTZ joined at the singularity, but it is not a true embedding at $X^0 = 0$: the tangent space map is not injective (the BTZ central singularity is a conical singularity).
- Combining the constraint equations we have $X^\mu X^\nu \eta^{(2,3)}_{\mu\nu} = -1 \Rightarrow$ embedding into $\widetilde{\mathrm{AdS}}_4$ exists.

- Analytic continuation $X^4 \to iX^4 \to$ Euclidean black hole $\subset M^{(1,4)}$.
- Also exists Euclidean solution with $(X^1, X^2) \leftrightarrow (X^3, X^4)$, $a \leftrightarrow 1/a$ *which has the same asymptotics* (c.f. Klein/Poincaré ball model, ideal boundaries coincide).
- $X^4 \to iX^4$ again $\to \mathrm{AdS}_3 \subset M^{2,4}$.

From this point of view, the analytic continuation gives $H \leftrightarrow \widetilde{\mathrm{AdS}}$ and EBTZ/thermal AdS $\leftrightarrow$ AdS!

Embedding in $\mathbb{M}_{3,2}$		Embedding in $\mathbb{M}_{4,1}$
$\widetilde{AdS_4}$ $-X_0^2+X_1^2+X_2^2+X_3^2-X_4^2=-1$		H_4 $-X_0^2+X_1^2+X_2^2+X_3^2+X_4^2=-1$
BTZ $X_1^2+X_2^2=\frac{a^2}{1+a^2}X_0^2,$ $X_3^2-X_4^2=\frac{1}{a^2+1}X_0^2-1,$ $X_0>0$ ($X_0=0$ singular).	$X_4\to iX_4$ $\longrightarrow$	EBTZ $X_1^2+X_2^2=\frac{1}{\alpha^2+1}X_0^2,$ $X_3^2+X_4^2=\frac{\alpha^2}{1+\alpha^2}X_0^2-1,$ $X_0>0.$
		$(X_1,X_2)\to(X_3,X_4)$, $a\to 1/a$ $\downarrow$
AdS$_3$ $X_1^2+X_2^2=\frac{a^2}{1+a^2}X_0^2-1,$ $X_3^2-X_4^2=\frac{1}{a^2+1}X_0^2,$ $X_0>0$ (2 copies of AdS).	$X_4\to iX_4$ $\longleftarrow$	EBTZ/Thermal AdS $X_1^2+X_2^2=\frac{1}{\alpha^2+1}X_0^2-1,$ $X_3^2+X_4^2=\frac{\alpha^2}{1+\alpha^2}X_0^2,$ $X_0>0.$

Sections of an algebraic varietyety in complex hyperbolic space.

The AdS embedding generalises to arbitrary dimension:

Lemma

Let $\mathbb{M}_{n,2}$ be pseudo-Euclidean space with two time directions and standard coordinates $X^A = (T, S, X^1, ..., X^n)$ and α be a positive real number. Then the submanifold $\{X^A \in \mathbb{M}_{n,2} | X^A X^B \eta_{AB} = -1;\ (X^n)^2 = S^2 + \frac{\alpha^a}{1+\alpha^2} T^2;\ T, X^n > 0\}$ is homeomorphic to $\mathbb{R}^n$ and globally isometric to AdS_n.

Note we have a continuous family of embeddings parameterised by α, which are all isometric to each other.

Conformal ball model of hyperbolic space

Unit ball ($x \cdot x < 1$) with metric:

$$g = \frac{dx \cdot dx}{1 - x \cdot x} + \frac{(x \cdot dx)^2}{(1 - x \cdot x)^2}$$

Geodesics, totally geodesic surfaces and geodesic spheres coincide with Euclidean counterparts.

- The Euclidean BTZ is given by $x^2 + y^2 = a^2/(1 + a^2)$. This is a hypercylinder
- More generally [Nomizu 73] a hypercylinder over any plane curve in the Klein ball is locally isometric to hyperbolic space.
- It is fairly straightforward to check that a hypercylinder over any closed plane curve is globally isometric to the nonrotating BTZ! (the conformal metric at infinity is an untwisted torus. BTZ is unique hyperbolic manifold with this conformal boundary.)

Asymptotic conditions

We consider manifolds which tend to constant negative curvature at infinity.

Definition (Conformally compactifiable)

Let (M, g) be a complete noncompact Riemannian manifold. If the we can find:

1. A compact Riemannian manifold $(\hat{M}, \hat{g})$ such that the interior $\hat{M} \setminus \partial\hat{M}$ is diffeomorphic to M;
2. A defining function Ω which has a simple zero at $\partial\hat{M}$ and vanishes nowhere in the interior of $\hat{M}$;
3. A diffeomorphism $\phi : \hat{M} \setminus \partial\hat{M} \to M$ such that $\Omega^2 \phi^* g = \hat{g}$,

then we say that M is conformally compactifiable.

Definition (Locally asymptotically hyperbolic)

If furthermore:

4. the defining function satisfies $\hat{g}^{ab}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega = \kappa^2$ everywhere on $\partial\hat{M}$

then M is asymptotically locally hyperbolic.

(under Weyl transformation, the Riemann tensor transforms:

$$R^{ab}{}_{cd} = \Omega^2\tilde{R}^{ac}{}_{cd} - 4\Omega\delta^{[a}_{[c}\tilde{\nabla}_{d]}\tilde{\nabla}^{b]}\Omega - 2\delta^{[a}_{[c}\delta^{b]}_{d]}\tilde{\nabla}^e\Omega\tilde{\nabla}_e\Omega\,.$$

Condition 4) $\Rightarrow R^{ab}{}_{cd} \to -2\delta^{[a}_{[c}\delta^{b]}_{d]}\kappa^2$.)

A preliminary result:

Lemma

Let (M_n, g) be some noncompact Riemannian manifold and $K_m = (\mathbb{B}_m, g_K)$ be the Klein ball model for H^m for some $m > n$. If there exists a smooth isometric immersion $\psi : M_n \to \mathbb{B}_m$ such that:

1. *The closure $\overline{\psi(M)}$ is a smooth submanifold with boundary $\partial\psi(M) \subset \partial\mathbb{B}$;*
2. *The tangent space of $\overline{\psi(M)}$ is not a subspace of that of $\partial\mathbb{B}$ at the boundary,*

then (M_n, g) is asymptotically hyperbolic in the conformally compactifiable sense.

Outlook

- Classification of embeddings of locally AdS 3-manifolds.
- Embedding of asymptotic region. Applications in terms of holography.