GAUGE THEORIES OF GRAVITATION: COMMUTATIVE AND NONCOMMUTATIVE FORMULATIONS

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1. Commutative gauge theory

Consider SL(2,*C*) as gauge group of the gravitation

$\Sigma_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b],\ a,b = 0,1,2,3$ - Lie algebra generators

γ_a - Dirac matrices, $\{\gamma_a, \gamma_b\} = 2\eta_{ab},\ \eta_{ab} = diag(-1,1,1,1)$

$\omega = \frac{1}{2}\omega_\mu^{ab}\Sigma_{ab}\, dx^\mu = \omega_\mu dx^\mu$ - spin connection

$e = e_\mu^a \gamma_a\, dx^\mu = e_\mu\, dx^\mu,\ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ - tetrad gauge fields

$\lambda = \frac{1}{2}\lambda^{ab}\Sigma_{ab}$ - gauge parameter

$\delta_\lambda e = i[\lambda, e],\quad \delta_\lambda \omega = d\lambda + i[\omega, \lambda],\quad \delta_\lambda R = i[\lambda, R]$

$R = \frac{1}{2}R_{\mu\nu}^{ab}\Sigma_{ab}\, dx^\mu \wedge dx^\nu = d\omega + i\,\omega\wedge\omega$ - curvature

$T = T_{\mu\nu}^a \gamma_a\, dx^\mu \wedge dx^\nu = de + i(\omega\wedge e + e\wedge\omega)$ - torsion

Integral of action

$$S_g = \int Tr((c_0 + c_1\gamma_5) e\wedge e\wedge R + c_2\gamma_5\, e\wedge e\wedge e\wedge e) \quad \Leftrightarrow$$

$$S_g = \int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\, Tr((c_0 + c_1\gamma_5) e_\mu e_\nu R_{\rho\sigma} + c_2\gamma_5\, e_\mu e_\nu e_\rho e_\sigma)$$

With appropriate values of the arbitrary constants c_0, c_1 and c_2, one obtain the Einstein-Hilbert action plus a cosmological constant.

$\Gamma_{\mu\nu}^\rho = e_a^\rho(\omega_\nu^{ab} e_{b\mu} + \partial_\nu e_\mu^a)$ - non-symmetric connection

2. Nonommutative gauge theory

Consider $GL(2,C)$ as gauge group of NC gravitation

$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x)$ - guarantees Lorentz invariance

$\alpha * \beta = \alpha\wedge\beta + \sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n C_n(\alpha,\beta)$ - covariant star product

$$C_1(\alpha,\beta) \equiv \theta^{\mu\nu}\nabla_\mu\alpha^a \wedge \nabla_\nu\beta^b + (-1)^{|\alpha|}\tilde{R}_{\mu\nu}^{\rho\sigma}\wedge(i_\rho\alpha^a)\wedge(i_\sigma\beta^b)$$

$C_n(\alpha,\beta),\quad n\geq 2$ - depending on curvature and torsion

Generators and gauge fields

$$(\Sigma_{ab}, \gamma_5, I),\quad \hat{\omega}_\mu = \frac{1}{2}\hat{\omega}_\mu^{ab}\Sigma_{ab},\quad \hat{E}_\mu = \hat{e}_\mu^a\gamma_a + \hat{f}_\mu^a\gamma_5\gamma_a$$

$$\hat{A}_\mu = \hat{\omega}_\mu + \hat{a}_\mu I + i\hat{b}_\mu\gamma_5,\quad \hat{\Lambda} = \lambda + aI + ib\gamma_5$$

Gauge transformations

$$\hat{\delta}\hat{A} = d\hat{\Lambda} + i[\hat{A}, \hat{\Lambda}]_*,\quad \hat{A} = \hat{A}_\mu dx^\mu$$

$$\hat{\delta}\hat{E} = i[\hat{E}, \hat{\Lambda}]_*,\quad \hat{E} = \hat{E}_\mu dx^\mu$$

$$\hat{F} = d\hat{A} - \frac{i}{2}[\hat{A}, A]_*,\quad \hat{\delta}\hat{F} = i[\hat{F}, \hat{\Lambda}]_*$$

Integral of action

$$S = \int d^4x(\det\theta^{\mu\nu})^{-\frac{1}{2}}\varepsilon^{\mu\nu\rho\sigma} tr((\alpha_1 + \beta_1\gamma_5)\hat{E}_\mu * \hat{E}_\nu * \hat{F}_{\rho\sigma} + (\alpha_2 + \beta_2\gamma_5)\hat{E}_\mu * \hat{E}_\nu * \hat{E}_\rho * \hat{E}_\sigma).$$

gauge invariant under $GL(2,C)$

3. Covariant Seiberg-Witten map

Gauge equivalence relation

$$\hat{A}_\mu(A;\theta) + \hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu(A;\theta) = \hat{A}_\mu(A + \delta_\lambda A;\theta)$$

Expand noncommutative fields as power series in $\theta^{\mu\nu}$

$$\hat{\Lambda} = \lambda + \Lambda^{(1)} + \Lambda^{(2)} + \cdots + \Lambda^{(n)} + \cdots$$

$$\hat{A}_\mu = A_\mu + A_\mu^{(1)} + A_\mu^{(2)} + \cdots + A_\mu^{(n)} + \cdots$$

$$\hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}^{(1)} + \hat{F}_{\mu\nu}^{(2)} + \cdots + \hat{F}_{\mu\nu}^{(n)} + \cdots$$

In the first order in $\theta^{\mu\nu}$

$$\Lambda^{(1)} = -\frac{1}{4}\theta^{\rho\sigma}\{A_\rho, \nabla_\sigma\lambda\}$$

$$A_\mu^{(1)} = -\frac{1}{4}\theta^{\rho\sigma}\{A_\rho, \nabla_\sigma A_\mu + F_{\sigma\mu}\},\quad \nabla_\sigma A_\mu = \partial_\sigma A_\mu - \Gamma_{\sigma\mu}^\rho A_\rho$$

$$F_{\mu\nu}^{(1)} = -\frac{1}{4}\theta^{\rho\sigma}(\{A_\rho, \nabla_\sigma F_{\mu\nu} + D_\sigma F_{\mu\nu}\} - 2\{F_{\mu\rho}, F_{\nu\sigma}\})$$

$$D_\sigma F_{\mu\nu} = \nabla_\sigma F_{\mu\nu} - i[A_\sigma, F_{\mu\nu}]$$

$$\nabla_\sigma F_{\mu\nu} = \partial_\sigma F_{\mu\nu} - \Gamma_{\sigma\mu}^\rho F_{\rho\nu} - \Gamma_{\sigma\nu}^\rho F_{\mu\rho}$$

Noncommutative curvature and torsion

$$\hat{F} = d\hat{A} - \frac{i}{2}[\hat{A}, \hat{A}]_*,\quad \hat{T} = d\hat{E} - i[\hat{E}, \hat{A}]_*$$

The first order of the torsion obtains with the expression

$$E_\mu^{(1)} = -\frac{1}{2}\theta^{\rho\sigma}\left\{A_\rho, \nabla_\sigma E_\mu + \frac{i}{2}[E_\mu, A_\sigma]\right\}$$

Commutative gravitation $\Rightarrow$ NC gravitation

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